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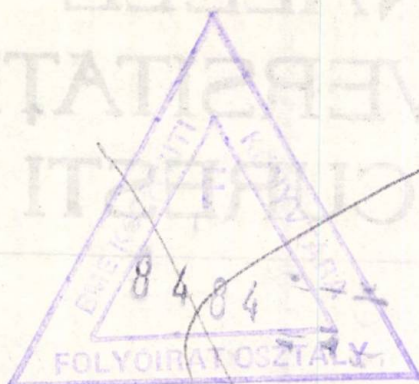
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ANNALE UNIVERSITATII BUDAPESTI



MATEMATICA

1883



REPRESENTATION OF (NO)-CONTINUOUS LINEAR OPERATORS ON THE SPACE $L_\mu(T, X)$

ISMAT BEG

In this article a theorem stating general form of (no)-continuous positive linear operators on the space of vector functions, integrable with respect to a set additive real function is proved. Terminology and notations used are that of Cristescu [2].

1. The space $L_\mu(T, X)$ of all μ -integrable functions

Let $\mathcal{T}$ be an algebra of subsets of a non-empty set V and μ is a positive, additive and real valued function defined on T . Suppose $\mathcal{X}$ is a σ -regular, σ -complete vector lattice. A function $f: T \rightarrow \mathcal{X}$ is simple, if

$$f(t) = \sum_{i=1}^{\infty} \chi_{A_i}(t) x_i \quad (1)$$

where $x_i \in \mathcal{X}$, $A_i \in \mathcal{T}$ but χ_{A_i} is characteristic function of set A_i .

With natural linear operations $L_\mu(T, X)$ is a vector space. Together with order relation $f \leq g$ if $f(t) \leq g(t)$ for every $t \in T$, the space $L_\mu(T, X)$ becomes a vector lattice, but $\|f\| = \int_T |f(t)| d\mu(t)$ is a vector semi-

norm on $L_\mu(T, X)$ (with values in $\mathcal{X}$) and $|f| \leq |g|$ implies $\|f\| \leq \|g\|$ (for details see Cristescu [1]).

Let $\mathcal{Y}$ be a complete vector lattice and m a positive, set additive function defined on $\mathcal{T}$ with values in $(\mathcal{X}, \mathcal{Y})_r$ the space of all regular (o)-continuous operators, which maps $\mathcal{X}$ into $\mathcal{Y}$.

If f is a simple function and given by the formula (1), by defining

$$\int_T \langle f(t), dm(t) \rangle = \sum_{i=1}^{\infty} \langle x_i, m(A_i) \rangle = \sum_{i=1}^{\infty} m(A_i)(x_i). \quad (2)$$

Then

$$\|f\| = \int_T |f(t)|, dm(t)$$

is also a vector seminorm with values in $\mathcal{Y}$ on the space of simple functions and

$$\left| \int_T \langle f(t), dm(t) \rangle \right| \leq \|f\|. \quad (3)$$

Using a result of Kawai [5], we can prove easily, that if Z is a regular, (LF)-lattice then any sequence $\{x_n\}$ of elements of Z is (o)-convergent, if $(o)\text{-}\lim_{n \rightarrow \infty} (x_n - x_{k_n}) = 0$ for every subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$.

2. (no)-continuous linear operators on the space $(L_\mu(T, X), \|\cdot\|)$

In this section, we shall denote by $\mathcal{X}$, σ -regular, complete vector lattice and by $\mathcal{Y}$, σ -regular, (LF)-complete vector lattice, respectively. O_1 and O_2 denotes origin of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Proposition 1. If the function m has property :

$$(o)\text{-}\lim_{n \rightarrow \infty} \|f_n\| = O_1 \text{ implies } (o)\text{-}\lim_{n \rightarrow \infty} \|f_n\| = O_2, \quad (4)$$

where f_n is any simple function, then integration (2) holds for any $f \in L_\mu(T, \mathcal{X})$ and it retains linearity and positivity.

Proof. Let f is any element of $L_\mu(T, X)$ and $\{f_n\}_{n \in \mathbb{N}}$ a μ -integrant sequence for f . Using property IV of integration from Cristescu [1], we have $(o)\text{-}\lim_{n \rightarrow \infty} \|f_n - f\| = O_1$. This implies $(o)\text{-}\lim_{k \rightarrow \infty} \|f_k - f_{n_k}\| = O_1$ for every subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ of $\{f_n\}_{n \in \mathbb{N}}$. By condition (4), we have $(o)\text{-}\lim_{k \rightarrow \infty} \|f_k - f_{n_k}\| = O_2$ for every subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$. Then

$$\left| \int_T \langle (f_k - f_{n_k})(t), dm(t) \rangle \right| \leq \|f_k - f_{n_k}\| \xrightarrow{o} O_2$$

this implies

$$(o)\text{-}\lim_{k \rightarrow \infty} \left| \int_T \langle f_k(t), dm(t) \rangle - \int_T \langle f_{n_k}(t), dm(t) \rangle \right| = O_2.$$

Now using the remark made at the end of section (1) about a regular, (LF) lattice, we obtained that the sequence $\left\{ \int_T \langle f_n(t), dm(t) \rangle \right\}_{n \in \mathbb{N}}$ is

(o)-convergent and there exists $(o)\text{-}\lim_{n \rightarrow \infty} \int_T \langle f_n(t), dm(t) \rangle$.

We shall denote

$$\int_T \langle f(t), dm(t) \rangle = (o)\text{-}\lim_{n \rightarrow \infty} \int_T \langle f_n(t), dm(t) \rangle. \quad (5)$$

Uniqueness: Suppose $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ are two different μ -integrant sequences for f . Then $(o)\text{-}\lim \|\|f_n - f\|\| = O$, and $(o)\text{-}\lim \|\|g_n - f\|\| = O_1$. This implies $(o)\text{-}\lim \|\|f_n - g_n\|\| = O_1$. By condition (4), this means $(o)\text{-}\lim \|\|f_n - g_n\|\| = O_2$. This implies

$$\left| \int_T \langle f_n(t), dm(t) \rangle - \int_T \langle g_n(t), dm(t) \rangle \right| \xrightarrow{o} O_2.$$

This shows that the limit (5) is independent of the sequence chosen $\{f\}_{n \in \mathbb{N}}$.

Obviously, integral (5) is a positive linear operator defined on $L_\mu(T, X)$.

Proposition 2. If m is a positive, additive function defined on $\mathcal{T}$ with values in the space $(\mathcal{X}, \mathcal{Y})_0^r$, satisfying the condition (4), then the formula

$$U(f) = \int_T \langle f(t), dm(t) \rangle$$

defines a (no)-continuous positive linear operator on $L_\mu(T, X)$ with values in $\mathcal{Y}$.

Proof. We need here to prove that

$$(o)\text{-}\lim_{n \rightarrow \infty} \|\|f_n\|\| = O_1 \text{ implies } (o)\text{-}\lim_{n \rightarrow \infty} U(f_n) = O_2$$

for any $f_n \in L_\mu(T, \mathcal{X})$.

This is obviously true, when f_n are simple functions. Otherwise, when f_n is not a simple function, let $\{g_n^k\}_{k \in \mathbb{N}}$ is a sequence of simple functions, μ -integrant for f_n .

As the space $\mathcal{Y}$ is σ -regular, let r is a common convergence regulator for the sequences $\{U(g_n^k)\}_{k \in \mathbb{N}}$ ($n = 1, 2, \dots$), which are convergent to $U(f_n)$ respectively. There exists $j_n \in \mathbb{N}$ such that for all $k \geq j_n$, we have

$$|U(g_n^k) - U(f_n)| \leq \frac{1}{n} r. \quad (6)$$

But $(o)\text{-}\lim_k \|\|g_n^k\|\| = \|\|f_n\|\|$, and the space $\mathcal{X}$ is regular, thus there exist an increasing sequence of natural numbers $\{k_n\}_{n \in \mathbb{N}}$ such that

$$(o)\text{-}\lim_{n \rightarrow \infty} \|\|g_{k_n}^{k_n}\|\| = (o)\text{-}\lim_{n \rightarrow \infty} \|\|f_n\|\| = O_1;$$

we can suppose that $j_n \leq k_n$.

Now let r' be a convergence regulator for the sequence $\{\|g_n^{k_n}\|\}_{n \in \mathbb{N}}$ which is convergent to O_2 . Thus there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of positive real numbers convergent to O such that for all $n \in \mathbb{N}$

$$\|g_n^{k_n}\| \leq \lambda_n r'. \quad (7)$$

Now we have

$$|U(f_n)| \leq |U(f_n) - U(g_n^{k_n})| + |U(g_n^{k_n})|$$

using (3), (6) and (7)

$$|U(f_n)| \leq \frac{1}{n} r + \lambda_n r'.$$

This implies that the sequence $\{U(f_n)\}_{n \in \mathbb{N}}$ is convergent to O_2 and this completes the proof.

Proposition 3. If U is a (no)-continuous positive linear operator defined on $L_\mu(T, X)$ with values in $\mathcal{Y}$, then there exist a positive, additive function m on $\mathcal{T}$ with values in $(\mathcal{X}, \mathcal{Y})_o$, satisfying condition (4) and such that

$$U(f) = \int_T \langle f(t), dm(t) \rangle \quad (8)$$

for any $f \in L_\mu(T, X)$

Proof. If for every $A \in \mathcal{T}$ and $x \in X$, we put

$$m(A)(x) = U(\chi_A x)$$

then obviously the operator $m(A)$ defined on X with values in $\mathcal{Y}$ is a positive, linear operator.

Now suppose $(o)\text{-}\lim_j \|\chi_{A_j} x\| = O_1$, this implies $(o)\text{-}\lim U(\chi_{A_j} x) = O_2$, this implies that $(o)\text{-}\lim_j \|\chi_{A_j} x\| = O_2$. Thus m satisfy relation (4). Evidently m is positive, additive function.

Let $f \in L_\mu(T, X)$. If f is a simple function, then

$$\begin{aligned} U(f) &= U\left(\sum_{i=1}^n \chi_{A_i} x_i\right) \\ &= \sum_{i=1}^n m(A_i)(x_i) \\ &= \int_T \langle f(t), dm(t) \rangle \end{aligned}$$

In the case, when $f \in L_\mu(T, \mathcal{X})$ is not a simple function, let $\{f_v\}_{v \in \mathbb{N}}$ is a μ -integrant sequence for f . Then with property (IV) of integration from Cristescu [1], we have

$$(o)\text{-}\lim_v \int_T |f_v(t) - f(t)| d\mu(t) = O_1$$

this implies $(o)\text{-}\lim_v \|f_v - f\| = O_1$. Thus $(o)\text{-}\lim_v U(f_v - f) = O_2$. Therefore $U(f) = (o)\text{-}\lim_v U(f_v) = (o)\text{-}\lim_v \int_T \langle f_v(t), dm(t) \rangle$.

On the other hand, for $i, x \geq v$,

$$\begin{aligned} & (o)\text{-}\lim_v \sup_{i, x \geq v} \|f_i - f_x\| \\ &= (o)\text{-}\lim_v \sup_{i, x \geq v} \int_T |f_i(t) - f_x(t)| d\mu(t) = O_1. \end{aligned}$$

As U is (no)-continuous therefore,

$$(o)\text{-}\lim_v \sup_{i, x \geq v} U(|f_i - f_x|) = O_2. \quad (9)$$

Relation (4) implies that the sequence $\{f_v\}_{v \in \mathbb{N}}$ is m -almost uniformly convergent to f and as

$$\begin{aligned} & (o)\text{-}\lim_v \sup_{i, x \geq v} \int_T \langle |f_i(t) - f_x(t)|, dm(t) \rangle \\ &= (o)\text{-}\lim_v \sup_{i, x \geq v} U(|f_i - f_x|) = O_2. \end{aligned}$$

Therefore, $\{f_v\}_{v \in \mathbb{N}}$ is a m -integrant sequence for f . Thus

$$U(f) = (o)\text{-}\lim_v U(f_v) = \int_T \langle f(t), dm(t) \rangle.$$

Proposition (2) and (3) implies the following theorem.

Theorem 4. The general form of the (no)-continuous, positive, linear operators, defined on $L_\mu(T, X)$ and taking values in $\mathcal{Y}$, is given by the formula

$$U(f) = \int_T \langle f(t), dm(t) \rangle$$

where m is a positive, additive function on $\mathcal{T}$, taking values in $(\mathcal{X}, \mathcal{Y})_0^r$ and satisfying condition (4).

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SPACE — TIME SINGULARITIES*

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A descriptive account is given of the background to current geometrical work on space time singularities through the completion of principal bundles by means of a Riemannian structure induced by the Levi-Civita connection. The original process used the orthonormal bundle and this admits useful modifications by taking a projective limit of completions or by invoking a parallelization. Analogous procedures are available on the frame bundle of order two, which may have more physical significance because these contain the original completions and are intrinsically sensitive to the acceleration of curves to which endpoints are supplied.

Introduction

One of the most intriguing results of modern classical physics is that any reasonable theory of gravity predicts the existence of physical singularities. Neither symmetric geometries nor particular theories, like relativity are required; singularities arise from the universal attractive phenomenon for matter. Perhaps as many believe, escape from singularities can be achieved by a quantum fine structure in the geometry. At the present there is no consensus among the experts as to which theory of quantum gravity is most natural physically. However, it is a notable result of one of these theories, geometric quantization, that no escape from gravitational collapse is possible even at the quantum level.

There is a large body of data from observational astronomy that supports the existence of singularities to the extent that other interpretations of the data are less natural. Indeed, most cosmologists believe that there was an initial singularity, the Big Bang some 10^{10} years ago, and that there may exist fully collapsed stars in the form of black holes, perhaps near Cygnus XI in particular.

Geometry, though it had its origin in studies of the surface of the Earth as its name suggests, was quickly put to work in a much larger arena than terrestrial navigation. But it was many centuries before its role in celestial studies was recognised as a mathematical model for gravity itself. Gauss said that mathematics is the queen of science, he would surely have agreed that her most trusted counsellor is geometry.

In our century, theoretical physics and geometry have developed enormously and much mutual benefit has been gained through their interplay. On the one hand we have quantum theory and its close association with the infinite dimensional geometry of functional analysis. On the other hand we have general relativity and its interaction with non-Euclidian

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geometry of finite dimensions. Both have been influential in the development of the geometrical superstructure of bundle theory, which is now their common language. The inevitable process of abstraction can sometimes leave the impression that manipulative calculus and algebraic detail are at the centre but these are in fact tools for preparing a geometrical view of physical phenomena. In relativity this is even more important to keep sight of because the essential geometry is Lorentzian not Riemannian and our intuition is easily led astray if we become remote from the physical meaning of our constructions.

Singularities in Riemannian manifolds can be handled and visualised intuitively fairly safely because a powerful theorem shows that any reasonable criterion for incompleteness corresponds precisely with the situation in metric topology. Thus, if we are able to use the space $\mathbb{Q}^4$ for physics then we would view $\mathbb{R}^4$ as its completion and $\mathbb{R}^4 \setminus \mathbb{Q}^4$ as the boundary set of singularities. In fact, we are obliged to use a 4-manifold modelled on $\mathbb{R}^4$ with a Lorentz structure determined implicitly by the disposition of matter. Currently, the presence of a singularity is inferred from the existence of a curve that is inextendible topologically but finite in some physically meaningful sense. A survey of the details can be found in [2].

First frame bundle completion

A spacetime (M, g) is a Hausdorff connected, noncompact, maximally extended, time and space oriented, smooth Lorentz 4-dimensional manifold. The standard reference for our purpose is Hawking and Ellis [9]; background geometry and motivation can be found in Dodson and Poston and further mathematical material is available in Sachs and Wu [12].

Since we have a smooth Lorentz structure and M is Hausdorff it follows that the topology is paracompact and hence metrizable through a partition of unity. This seems to be an advantage because the metrization is achieved through a Riemannian structure subordinate to the partition of unity. For a Riemannian manifold is a metric space and so the criterion for completeness is that of Cauchy sequences, hence a completion metric space can be constructed by standard procedure. Unfortunately there is no information available to help in our choice of partition of unity for the exploitation of paracompactness. However, in Riemannian geometry there is a very nice theorem which shows that Cauchy completeness is equivalent to geodesic completeness. Our Lorentz structure determines the Levi-Civita connection so geodesics are well defined in space time and following Einstein, we interpret them, if null or time like, as the possible trajectories of particles that are free of external constraints except gravity. Geodesics have the great advantage of possessing a distinguished family of affine parameters and finiteness of affine parameter length is invariant under changes of affine parameter. So the notion of geodesics completeness for spacetimes could yield a useful criterion. Unfortunately, two problems arise. Firstly, there seems to be no natural choice of equivalence relation on geodesics that allows the construction of a boundary

for spacetime in a reasonable way, as the set of endpoints for affinely finite inextensible geodesics in M . Secondly, there exist spacetimes without finite inextensible geodesics but with finite inextensible timelike curve that could represent the trajectory of a particle of matter subjected to finite external forces.

This situation led Schmidt [13] to invent another criterion for finiteness of an inextensible curve in M . He used the basic 1-form θ and the connection form ω of the Levi Civita connection ∇ to put a Riemannian structure g_1 on the frame bundle LM , or more precisely on a component of the orthonormal bundle. In this metric the finiteness of the horizontal lift of a curve in M is invariant under the structure group. So singularities can be defined in this way, but more importantly, it is possible to incorporate them as points in a topological space $\bar{M}$ in which M is dense; $\bar{M}$ is the quotient by the structure group of the Cauchy completion of the frame bundle. This is well defined because the action of the group is uniformly continuous and so extends uniquely to the points of completion. The complement $\partial M = \bar{M} \setminus M$ is called the b -boundary of (M, g) , it consists of the topological endpoints of inextensible curves in M that have finite horizontal lift in Schmidt's metric.

The process is nice mathematical because it gives the classical answer when applied to a Riemannian manifold and it is nice physically because it makes essential and intrinsic use of the connection. This latter point is important in view of our earlier remarks, because the details of the b -completion process in a particular example can be quite formidable. Physically, the connection has a more fundamental role than the metric itself. Firstly, the distribution of matter determines primarily the connection, and secondly the connection is in principle more directly observable by careful measurement of trajectories of free test particles.

There are, however, some snags. Though we could not reasonably expect M to be a manifold, it turns out that it is not necessarily even a Hausdorff space. In particular ∂M is not Hausdorff separated from M , which poses problems when trying to investigate physical process „near” to the singularity. Also, some points in ∂M can arise from curves trapped in a compact set. In fact for the (non-Riemannian) geometry of S^1 with constant connection, ∂S^1 consists of a single point outside S^1 and not Hausdorff separated from any other point. Similar problems arise in Friedmann spacetime which is an acceptable cosmological model. So we turn to some modifications of the original Schmidt process.

Parallelization completion

If we have a smooth parallelization $p: M \rightarrow LM$ then it determines a flat connection $\bar{\nabla}$ with connection form $\bar{\omega}$. This was used [2] to add a further term to Schmidt's frame bundle metric g_1 to give a new Riemannian structure, $\bar{g}_1$. The motivation came from examples like S^1 and Friedmann spacetime where unphysical topological identifications arose from horizontal curves that could in g_1 be made arbitrarily short by sufficiently extreme lifting. This is avoided in $\bar{g}_1$ but in other respects it beha-

ves as well as g_1 and allows a similar construction of p -boundary and a p -completion $\bar{M}^p$ in which M is dense. The nice mathematical result here [7] is that $\bar{M}^p$ is homeomorphic to the Cauchy completion of the Riemannian submanifold pM in (LM, g_1) and it is therefore necessarily Hausdorff.

Now there are physical snags. For though most realistic spacetimes are parallelizable it is in general not clear which particular parallelizations have real significance. Motivation for a choice depends on further information, such as a distinguished matter field, for example. However, the obvious choice of parallelization in Friedmann spacetime appears to give a physically reasonable completion space [2].

Projective limit completion

The next development in bundle completion was due to Clarke [1] and was specifically designed to avoid the poor separation of ∂M from M and to eliminate boundary points arising from curves trapped in a compact set.

This construction is essentially categorical. The projective limit, or left limit, of a diagram in Top indexed by some set J ,

$$\Delta = \{A_i \xrightarrow{P_{ij}} A_j \mid i, j \in J\}'$$

is a topological space L and continuous maps f with

$$\varprojlim \Delta = \{L \xrightarrow{f_i} A_i \mid i \in J\}$$

satisfying the universal property that for all diagrams

$$\{K \xrightarrow{h_i} A_i \mid i \in J\}$$

in Top there is a unique $b: K \rightarrow L$ in Top with

$$g_i \circ h_i = b \text{ for all } i \text{ in } J.$$

Clarke began with the partially ordered inclusions of all cocompact open sets of M . Each member set is then b -completed as a submanifold of M and joined to a copy of M through identification of interior points. This yields another diagram in Top and its left limit is the projective limit completion space $\bar{M}$ for (M, g) .

By construction, $\bar{M}$ consists of generalised sequences drawn from the constituent objects of the diagram whose limit was taken. The set M° of constant sequences from M is in fact homeomorphic to M itself and

also is dense in $\overleftarrow{M}$; therefore we do indeed have a projective limit boundary $\partial\overleftarrow{M} = \overleftarrow{M} \setminus M^0$. This boundary is Hausdorff separated from M^0 because a continuous curve in M^0 trapped in a compact set cannot be continuously extended to $\partial\overleftarrow{M}$.

In this way the unwanted identification of initial and final singularities in the closed Friedmann spacetime is avoided, but only for the case of positive curvature. In the case of negative curvature it is not possible to remove a compact set that disconnects the early times from the late times, so the identification remains.

Second frame bundle completion

The final process that we consider here is in fact a natural generalisation of the Schmidt process; rather than a modification. It arose from discussions at a conference following the presentation of a paper by Radioioivici [11] on a globalization of some results of Yano and Ishihara [14]. Its intrinsic value stems from an essential involvement of acceleration structure in the synthesis of a boundary for spacetime and we should consider why this is so.

Given a curve c in M , representing the trajectory of a particle with velocity vector c , the covariant acceleration is precisely the obstruction to the curve developing as a geodesic. For the particle, this obstruction is physically felt as an external force constraining its freedom. It is possible to construct a bundle of curves equivalent up to covariant acceleration through the points of spacetime. In the presence of a connection this bundle $T^{(2)}M$ is a vector bundle with fibre $\mathbb{R}^8$ which generalizes in a natural way the tangent bundle TM . For TM can be viewed as the bundle of curves equivalent up to velocity through the points of M .

From $T^{(2)}M$ one obtains a principal bundle $L^{(2)}M$ [4], the frame bundle of order two, in which the connection ∇ in LM induces a connection $\tilde{\nabla}$. An analogous method to that used for the first frame bundle can be used to give a Riemannian structure g_2 on $L^{(2)}M$, from the basic 1-form $\tilde{\theta}$ and connection form $\tilde{\omega}$. Again the structure group acts uniformly continuously in the metric g_2 and so determines a quotient topological space $\tilde{M}^2$ from the Cauchy completion of a component of $(L^{(2)}M, g_2)$. Also M is dense in $\tilde{M}$ so we do again obtain a boundary for (M, g) , the $\tilde{b}$ -boundary:

$$\partial\tilde{M} = \tilde{M} \setminus M.$$

Now $\tilde{M}$ is not a Hausdorff space in general but it is known [5] that $\partial\tilde{M}$ contains ∂M and presumably its intrinsic sensitivity to the acceleration of curves to which it supplies endpoints has physical significance. In order to distinguish between $\partial\tilde{M}$ and ∂M , it would be useful to strengthen the result in [8] that an inextensible timelike curve c with unit

tangent vector $\dot{c}$ and finite proper time t_m ends on ∂M if :

$$\int_0^{t_m} \exp \left(\int_0^t \|\nabla_{\dot{c}} \dot{c}\| ds \right) dt < \infty$$

Thus we are interested for example in those curves, if any exists which end on ∂M but for which this integral is infinite.

Just as the b -completion can be modified by a parallelization $p:M \rightarrow LM$ so can the $\tilde{b}$ -completion by means of a section $p:M \rightarrow L^{(2)}M$, with corresponding Hausdorff $\tilde{p}$ -completion.

Another second order structure may be useful in the investigation of spacetime singularities. This structure is $P^{(2)}M$, the Ehresmann bundle of 2-frames, to which the tangent 2-jet bundle is associated [10]. A connection in LM induces a section of $P^{(2)}M$ over $GL(4; R)$ and hence a frame field for pure second order tangent vectors [6].

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ON SOME INFORMATIONAL PROPERTIES OF BAYESIAN RANDOM EVOLUTION

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The Bayesian approach in mathematical statistics is much utilized as it leads to interesting results, especially in the field of estimation theory. At the same time, Bayesian random evolution is a particular case of the weighting process studied in the information theory, for which was established the converse H-theorem.

The paper deals with the Bayesian random evolution on an arbitrary set, using some tools of the information theory. Continuous entropy leads to a property which is analogous to the „information balance” for Shannon entropy. Using the discrimination function introduced by S. Kullback, we obtain a characterization of Bayes a posteriori probability.

1. Bayesian random evolution

Let us consider a statistical population $(\Omega, \mathcal{K})$ and a parameter space $(\Theta, \mathcal{A})$, where Θ is an open set in $\mathbb{R}^k$. We suppose that $\mathcal{A}$ is a σ -algebra generated by $\mathcal{C}$, $\mathcal{A} = \sigma(\mathcal{C})$, which has the following property: $(C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 \in \mathcal{C})$.

Let $X: (\Omega, \mathcal{K}) \rightarrow (S, \mathcal{S})$ be the random characteristic of the population and $(X_1, \dots, X_n): (\Omega, \mathcal{K}) \rightarrow (S^n, \mathcal{S}^n)$ be the n -dimensional selection vector. We suppose that $S \subset R$, $\mathcal{S} = \sigma(\mathcal{G})$, $\mathcal{S}^n = \sigma(\mathcal{G}^n)$, where $\mathcal{G}$ has the same property as $\mathcal{C}$.

The a priori probability on $(\Theta, \mathcal{A})$ is absolutely continuous with respect to a probability λ :

$$\mu: \mathcal{A} \rightarrow [0, 1], \mu \ll \lambda, \mu(A) = \int_A \rho(\theta) d\lambda(\theta), A \in \mathcal{A}.$$

In any problem of statistical inference we have a family of probabilities on $(\Omega, \mathcal{K})$ depending on the θ -parameter, $\theta \in \Theta$. Let these probabilities be continuous functions in θ . It means that $P(\circ|\theta): \mathcal{K} \times \Theta \rightarrow [0, 1]$ is a transition probability.

X and $(X_1, \dots, X_n)$ induce transition probabilities

$$P(\circ|\theta) \circ X^{-1} = F(\circ|\theta): \mathcal{S} \times \Theta \rightarrow [0, 1],$$

$$P(\circ|\theta) \circ (X_1, \dots, X_n)^{-1} = F_n(\circ|\theta): \mathcal{S}^n \times \Theta \rightarrow [0, 1].$$

We suppose that there is a probability η on $(S, \mathcal{S})$ so that $F(\circ|\theta) \ll \eta$, $F_n(\circ|\theta) \ll \eta^n$ for any $\theta \in \Theta$. Let us denote $p(x|\theta) = \frac{dF(\circ|\theta)}{d\eta}$, $p_n(x_1, \dots, x_n|\theta) = \frac{dF_n(\circ|\theta)}{d\eta^n}$.

We introduce the following probabilities on $(\Theta \times S^n, \mathcal{A} \times \mathcal{S}^n)$ and on $(S^n, \mathcal{S}^n)$:

$Q_n: \mathcal{A} \times \mathcal{S}^n \rightarrow [0, 1]$, where $\mathcal{A} \times \mathcal{S}^n = \sigma(\mathcal{C} \times \mathcal{G}^n)$ and for any $A \in \mathcal{C}$ and $\tilde{M} \in \mathcal{G}^n$ we have $Q_n(A \times \tilde{M}) = \int_A F_n(\tilde{M} | \theta) d\mu(\theta)$;

$P_n: \mathcal{S}^n \rightarrow [0, 1]$ defined for any $\tilde{M} \in \mathcal{G}^n$ by $P_n(\tilde{M}) = Q_n(\Theta \times \tilde{M})$. Since the conditions of Fubini theorem are satisfied, we get

$$\begin{aligned} P_n(\tilde{M}) &= \int_{\Theta} \int_{\tilde{M}} p_n(x_1, \dots, x_n | \theta) \rho(\theta) d\eta^n(x_1, \dots, x_n) d\lambda(\theta) = \\ &= \int_{\tilde{M}} q_n(x_1, \dots, x_n) d\eta^n(x_1, \dots, x_n), \text{ where} \end{aligned}$$

$$q_n(x_1, \dots, x_n) = \int_{\Theta} p_n(x_1, \dots, x_n | \theta) \rho(\theta) d\lambda(\theta).$$

Notice. The fact that the sample variables $X_1, \dots, X_n$ are independent with respect to $P(\circ | \theta)$ for any θ does not imply their independence with respect to the density function $q_n(x_1, \dots, x_n)$.

The Bayes a posteriori probability is given by the transition probability

$$\mu_n: \mathcal{A} \times S_n \rightarrow [0, 1],$$

so that for any $A \in \mathcal{C}$, $\tilde{M} \in \mathcal{G}^n$ we have

$$Q_n(A \times \tilde{M}) = \int_{\tilde{M}} \mu_n(A | x_1, \dots, x_n) q_n(x_1, \dots, x_n) d\eta^n(x_1, \dots, x_n).$$

We take

$$\mu_n(A | x_1, \dots, x_n) = \int_A S_n(\theta | x_1, \dots, x_n) d\lambda(\theta) \quad (1)$$

with

$$\rho_n(\theta | x_1, \dots, x_n) = \frac{\rho(\theta) p_n(x_1, \dots, x_n | \theta)}{\int_{\Theta} \rho(\theta) p_n(x_1, \dots, x_n | \theta) d\lambda(\theta)} = \frac{\rho(\theta) p_n(x_1, \dots, x_n | \theta)}{q_n(x_1, \dots, x_n)}$$

Definition. The family $\{\mu_n(\circ | x_1, \dots, x_n) | n \in \mathbb{N}\}$ with μ_n given by relation (1) is called *Bayesian random evolution on Θ with a priori probability μ and the transition probability $P(\circ | \theta)$* .

Taking into account the analogy with the process of transmission of information, we notice that Bayes formula is a rule of inversion of the channel $\{\Theta, F_n(\cdot|\theta), S^n\}$.

2. Entropy. Entropic balance

The continuous entropy for a probability density function on an open and bounded set in $\mathbb{R}^n$ was interpreted as being (up to an additive constant) the variation of information when we pass from the initial uniform probability distribution to the new probability measure defined by the considered density function. If we want to have this interpretation of the continuous entropy, we have to consider Θ as a bounded set in $\mathbb{R}^k$ and S as a bounded interval in $\mathbb{R}$.

We associate the following entropies to the probability distributions considered above :

$$\begin{aligned}
 H(\Theta) &= - \int_{\Theta} \rho(\theta) \ln \rho(\theta) d\lambda(\theta) \\
 H(S^n|\theta) &= - \int_{S^n} p_n(x_1, \dots, x_n|\theta) \ln p_n(x_1, \dots, x_n|\theta) d\gamma^n(x_1, \dots, x_n) \\
 H(S^n|\Theta) &= \int_{\Theta} H(S^n|\theta) \rho(\theta) d\lambda(\theta) = \\
 &= - \int_{\Theta} \int_{S^n} p_n(x_1, \dots, x_n|\theta) \rho(\theta) \ln p_n(x_1, \dots, x_n|\theta) d\gamma^n(x_1, \dots, x_n) d\lambda(\theta) \\
 H(\Theta \times S^n) &= - \int_{\Theta} \int_{S^n} p_n(x_1, \dots, x_n|\theta) \rho(\theta) \ln (p_n(x_1, \dots, \\
 &\quad \dots x_n|\theta) \rho(\theta)) d\gamma^n(x_1, \dots, x_n) d\lambda(\theta) \\
 H(S_n) &= - \int_{S^n} q_n(x_1, \dots, x_n) \ln q_n(x_1, \dots, x_n) d\gamma^n(x_1, \dots, x_n) \\
 H(\Theta | x_1, \dots, x_n) &= - \int_{\Theta} \rho_n(\theta|x_1, \dots, x_n) \ln \rho_n(\theta|x_1, \dots, x_n) d\lambda(\theta) \\
 H(\Theta | S^n) &= \int_{S^n} H(\Theta|x_1, \dots, x_n) q_n(x_1, \dots, x_n) d\gamma^n(x_1, \dots, x_n) \\
 &= \int_{S^n} \int_{\Theta} \rho(\theta) p_n(x_1, \dots, x_n|\theta) \ln \rho_n(\theta|x_1, \dots, x_n) d\lambda(\theta) d\gamma^n(x_1, \dots, x_n).
 \end{aligned}$$



Proposition 2.1. $H(\Theta \times S^n) \leq H(S^n)$.

The property follows from Jensen inequality for a concave function Φ and a Lebesgue integrable function f on $(E, \mathcal{E}, P)$:

$$\Phi \left(\int_E f \, dP \right) \geq \int_E \Phi \circ f \, dP.$$

We take $\Phi(x) = -x \ln x$ for $x \geq 0$.

Proposition 2.2. $H(\Theta \times S^n) = H(\Theta) + H(S^n | \Theta) = H(S^n) + H(\Theta | S^n)$.

The property follows directly, using Fubini theorem.

Proposition 2.3. (Entropic balance)

$$H(\Theta) - H(\Theta | S^n) = H(S^n) - H(S^n | \Theta).$$

Let us denote for any $(x_1, \dots, x_n) \in S^n$ the likelihood function

$$L_n(\theta) = p_n(x_1, \dots, x_n | \theta), \text{ and } \Lambda_n(\theta) = \ln L_n(\theta).$$

The a posteriori expectation of $\Lambda_n(\theta)$ is denoted by

$$m(x_1, \dots, x_n) : m(x_1, \dots, x_n) = \int_{\Theta} \ln L_n(\theta) \, \rho_n(\theta | x_1, \dots, x_n) \, d\lambda(\theta).$$

Proposition 2.4. The expected value of $m(x_1, \dots, x_n)$ satisfies the equality:

$$\int_{S^n} m(x_1, \dots, x_n) q_n(x_1, \dots, x_n) \, d\eta^n(x_1, \dots, x_n) = -H(S^n | \Theta).$$

The property follows directly, using Fubini theorem.

3. The discrimination function of a posteriori and a priori probabilities on the parameter space

On the parameter space we have a priori and a posteriori probabilities given for any $A \in \mathcal{A}$ by:

$$\mu(A) = \int_A \rho(\theta) \, d\lambda(\theta)$$

$$\mu_n(A | x_1, \dots, x_n) = \int_A \rho(\theta | x_1, \dots, x_n) \, d\lambda(\theta) = \frac{\int_A \rho(\theta) p_n(x_1, \dots, x_n | \theta) \, d\lambda(\theta)}{\int_{\Theta} \rho(\theta) p_n(x_1, \dots, x_n | \theta) \, d\lambda(\theta)}$$

Their discrimination function is

$$I(\mu_n(\circ | x_1, \dots, x_n); \mu) = \int_{\Theta} \rho_n(\theta | x_1, \dots, x_n) \ln \frac{\rho_n(\theta | x_1, \dots, x_n)}{\rho(\theta)} d\lambda(\theta).$$

According to Kullback [4], we have

$$I(\mu_n(\circ | x_1, \dots, x_n); \mu) \geq 0 \text{ for any } (x_1, \dots, x_n) \in S^n.$$

Proposition 3.1. The expected value of the discrimination function

$$I(\mu_n; \mu) = \int_{S^n} I(\mu_n(\circ | x_1, \dots, x_n); \mu) q_n(x_1, \dots, x_n) d\eta^n(x_1, \dots, x_n)$$

verifies the following equality:

$$I(\mu_n; \mu) = H(S^n) - H(S^n | \Theta) = H(\Theta) - H(\Theta | S^n).$$

The property follows directly, using Fubini theorem.

Now let us suppose that the likelihood function $L_n(\theta)$ satisfies η^n -almost sure the properties

$$(a) (L_n(\theta))^{-\beta} \in \mathcal{L}_1(\mu), (\forall) \beta \in \mathbb{R};$$

$$(b) (L_n(\theta))^{-\beta} \ln L_n(\theta) \in \mathcal{L}_1(\mu), (\forall) \beta \in \mathbb{R}.$$

For $(x_1, \dots, x_n) \in S^n$, let us introduce a family of probabilities on $(\Theta, \mathcal{A})$ depending on $(x_1, \dots, x_n)$ and being absolutely continuous with respect to μ :

$$\{\nu_n(\circ | x_1, \dots, x_n) \mid \nu_n(\circ | x_1, \dots, x_n) \ll \mu\},$$

$$r_n(\theta; x_1, \dots, x_n) = \frac{d \nu_n(\circ | x_1, \dots, x_n)}{d\lambda}.$$

We suppose that

$$\begin{aligned} & I(\nu_n(\circ | x_1, \dots, x_n); \mu) = \\ & = \int_{\Theta} r_n(\theta; x_1, \dots, x_n) \ln \frac{r_n(\theta; x_1, \dots, x_n)}{\rho(\theta)} d\lambda(\theta) < \infty. \end{aligned}$$

Theorem 3.1. Let $(x_1, \dots, x_n) \in S^n$ be a n -dimensional sample. The probability density function

$$r_n(\theta; x_1, \dots, x_n) \geq 0, \int_{\Theta} r_n(\theta; x_1, \dots, x_n) d\lambda(\theta) = 1 \quad (\text{C.1})$$

which minimizes the discrimination function

$$I(v_n(\cdot | x_1, \dots, x_n); \mu) = \int_{\Theta} r_n(\theta; x_1, \dots, x_n) \ln \frac{r_n(\theta; x_1, \dots, x_n)}{\rho(\theta)} d\lambda(\theta)$$

subject to the constraint

$$\int_{\Theta} \ln L_n(\theta) r_n(\theta; x_1, \dots, x_n) d\lambda(\theta) = m(x_1, \dots, x_n) \quad (\text{C.2})$$

is given by

$$r_n^*(\theta; x_1, \dots, x_n) = \rho_n(\theta | x_1, \dots, x_n) = \frac{\rho(\theta) p_n(x_1, \dots, x_n | \theta)}{\int_{\Theta} \rho(\theta) p_n(x_1, \dots, x_n | \theta) d\lambda(\theta)}$$

Proof. We want to maximize the expression $-I(v_n(\cdot | x_1, \dots, x_n); \mu)$ subject to the constraints (C.1) and (C.2). Using Lagrange's multiplier method we get:

$$\begin{aligned} & -I(v_n(\cdot | x_1, \dots, x_n); \mu) - \alpha - \beta m(x_1, \dots, x_n) = \\ & = \int_{\Theta} r_n(\theta; x_1, \dots, x_n) \left(\ln \frac{\rho(\theta)}{r_n(\theta; x_1, \dots, x_n)} - \alpha - \beta \ln L_n(\theta) \right) d\lambda(\theta) \leq \\ & \leq \int_{\Theta} r_n(\theta; x_1, \dots, x_n) \left\{ \frac{(L_n(\theta))^{-\beta} \rho(\theta)}{r_n(\theta; x_1, \dots, x_n)} e^{-\alpha} - 1 \right\} d\lambda(\theta), \end{aligned}$$

where the equality holds if and only if

$$r_n(\theta; x_1, \dots, x_n) = (L_n(\theta))^{-\beta} \rho(\theta) e^{-\alpha \lambda} - \text{a.s.}$$

Introducing this last expression into the condition (C.1) we get:

$$\alpha = \ln \Phi(\beta), \text{ where}$$

$$\Phi(\beta) = \int_{\Theta} (L_n(\theta))^{-\beta} \rho(\theta) d\lambda(\theta).$$

Using the condition (C.2) we obtain the equation

$$\frac{d \ln \Phi(\beta)}{d\beta} = -m(x_1, \dots, x_n).$$

This equation has a unique solution $\beta^* = -1$. Then

$$r_n^*(\theta; x_1, \dots, x_n) = \frac{L_n(\theta) \rho(\theta)}{\int_{\Theta} L_n(\theta) \rho(\theta) d\lambda(\theta)} = \rho_n(\theta | x_1, \dots, x_n).$$

4. Infinit dimensional sample space

In this section we extend the notions which were presented in previous sections to the case of infinit dimensional sample space $(T, \mathcal{T})$ $T = S^\infty$, $\mathcal{T} = \sigma(\mathcal{G}^\infty)$.

The sample vector $X: (\Omega, \mathcal{H}) \rightarrow (T, \mathcal{T})$ induces the transition probability $\mathbb{F}(\cdot | \theta): \mathcal{T} \times \Theta \rightarrow [0, 1]$, $\mathbb{F}(\cdot | \theta) = P(\cdot | \theta) \circ \times^{-1}$. We suppose $\mathbb{F}(\cdot | \theta) \ll \gamma^\infty$ for any $\theta \in \Theta$, and let $p(y | \theta) = \frac{d\mathbb{F}(\cdot | \theta)}{d\gamma^\infty}$ be the probability density function.

Let us consider the product space $(\Theta \times T, \mathcal{A} \times \mathcal{T})$, where $\mathcal{A} \times \mathcal{T} = \sigma(\mathcal{C} \times \mathcal{G}^\infty)$ and the probability $\mathbb{Q}: \mathcal{A} \times \mathcal{T} \rightarrow [0, 1]$ given for any $A \in \mathcal{C}$ and $B \in \mathcal{G}^\infty$ by

$$\mathbb{Q}(A \times B) = \int_A \mathbb{F}(B | \theta) d\mu(\theta).$$

On $(T, \mathcal{T})$ we get the marginal distribution $\mathbb{P}: \mathcal{T} \rightarrow [0, 1]$ so that for any $B \in \mathcal{G}^\infty$, $\mathbb{P}(B) = \mathbb{Q}(\Theta \times B)$.

Using Fubini theorem we may write $\mathbb{P}(B) = \int_B q(y) d\gamma^\infty(y)$, where $q(y) = \int_{\Theta} p(y | \theta) \rho(\theta) d\lambda(\theta)$.

The a posteriori probability on $(\Theta, \mathcal{A})$ is given by the inverse transition probability $\mu: \mathcal{A} \times T \rightarrow [0, 1]$,

$$\mu(A | y) = \int_A \rho(\theta | y) d\lambda(\theta), \text{ where}$$

$$\rho(\theta | y) = \frac{\rho(\theta) p(y | \theta)}{\int_{\Theta} \rho(\theta) p(y | \theta) d\lambda(\theta)}.$$

Using these probability distributions we obtain the following entropies :

$$\begin{aligned}
 H(T|\Theta) &= - \int_T p(y|\theta) \ln p(y|\theta) d\gamma^\infty(y) \\
 H(T|\Theta) &= \int_\Theta H(T|\theta) \rho(\theta) d\lambda(\theta) = \\
 &= - \int_\Theta \int_T p(y|\theta) \rho(\theta) \ln p(y|\theta) d\gamma^\infty(y) d\lambda(\theta) \\
 H(\Theta \times T) &= - \int_\Theta \int_T p(y|\theta) \rho(\theta) \ln (p(y|\theta) \rho(\theta)) d\gamma^\infty(y) d\lambda(\theta) \\
 H(T) &= - \int_T q(y) \ln q(y) d\gamma^\infty(y) \\
 H(\Theta|y) &= - \int_\Theta \rho(\theta|y) \ln \rho(\theta|y) d\lambda(\theta) \\
 H(\Theta|T) &= \int_T H(\Theta|y) q(y) d\gamma^\infty(y) = \\
 &= - \int_T \int_\Theta \rho(\theta|y) q(y) \ln \rho(\theta|y) d\lambda(\theta) d\gamma^\infty(y)
 \end{aligned}$$

Proposition 4.1. The entropies constructed above using the infinitesimal sample space have the following properties :

- (a) $H(\Theta \times T) \leq H(T)$;
- (b) $H(\Theta \times T) = H(\Theta) + H(T|\Theta) = H(T) + H(\Theta|T)$;
- (c) *Entropic balance*: $H(\Theta) - H(\Theta|T) = H(T) - H(T|\Theta)$.

Now, we can construct the discrimination functions for the pairs of probabilities $(\mu(\cdot|y); \mu)$ and $(\mu(\cdot|y); \mu_n)$.

$$I(\mu(\cdot|y); \mu) = \int_\Theta \rho(\theta|y) \ln \frac{\rho(\theta|y)}{\rho(\theta)} d\lambda(\theta) \geq 0 \text{ for any } y \in T.$$

$$I(\mu; \mu) = \int_T I(\mu(\cdot|y); \mu) q(y) d\gamma^\infty(y).$$

Let us define

$$\mu_n(\circ | y) = \mu_n(\circ | x_1, \dots, x_n) \text{ for any } y = (x_1, \dots, x_n, \dots) \in T,$$

$$\rho_n(\theta | y) = \rho_n(\theta | x_1, \dots, x_n) \text{ for any } y = (x_1, \dots, x_n, \dots) \in T.$$

Then

$$I(\mu(\circ | y); \mu_n(\circ | y)) = \int_{\Theta} \rho(\theta | y) \ln \frac{\rho(\theta | y)}{\rho_n(\theta | y)} d\lambda(\theta) \geq 0$$

for any $y \in T$,

$$I(\mu; \mu_n) = \int_T I(\mu(\circ | y); \mu_n(\circ | y)) q(y) d\gamma^\infty(y).$$

Proposition 4.2. $I(\mu; \mu) = H(T) - H(T | \Theta) = H(\Theta) - H(\Theta | T) \geq 0$.

Proposition 4.3. $I(\mu; \mu) = I(\mu_n; \mu) + I(\mu; \mu_n)$.

The property follows from the evaluation of $I(\mu; \mu_n)$, using Fubini theorem.

Notice. If the a posteriori probability on $(\Theta, \mathcal{A})$ constructed using the infinit dimensional sample space is concentrated on the real value of the parameter $\{\theta_0\}$ we get $H(\Theta | y) = 0$ for any $y \in T$, and then $H(\Theta | T) = 0$, $I(\mu; \mu) = H(\Theta)$.

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OPÉRATEURS PSEUDO-DIFFÉRENTIELS ITÉRÉS ET ANALYTICITÉ

V. IFTIMIE

On applique certains résultats d'hypoellipticité partielle analytique de [5] pour prouver une extension du théorème de Kotake et Narasimhan [6] concernant les itérés des opérateurs différentiels elliptiques aux coefficients analytiques au cas des opérateurs pseudo-différentiels elliptiques analytiques de Trèves [8], d'ordre rationnel positif.

Soit Γ une variété analytique réelle, compacte et sans bord, de dimension finie n , où on a fixé une densité analytique. Si P est un opérateur pseudo-différentiel analytique sur Γ (au sens de Boutet de Monvel [2]) d'ordre $m \in \mathbb{N}^*$, Baouendi et Goulaouic [1] prouvent que pour toute fonction analytique u sur Γ , il existe une constante $C > 0$ telle que pour tout $k \in \mathbb{N}$,

$$(1) \quad \|P^k u\|_{L^2(\Gamma)} \leq C^{k+1} (k!)^m.$$

Réciproquement, Damakhi [3] prouve que si P est elliptique et son symbole principal est réel, alors toute fonction $u \in \mathcal{C}^\infty(\Gamma)$ qui vérifie les estimations (1) est analytique.

Supposons maintenant que P soit un opérateur pseudo-différentiel analytique au sens plus général de Trèves [8] d'ordre $m \in \mathbb{R}$. Plus exactement, on suppose que P est une application linéaire et continue de $\mathcal{C}^\infty(\Gamma)$ à $\mathcal{D}'(\Gamma)$, telle que pour toute carte analytique (Ω, χ) sur Γ (Ω identifié à un ouvert de $\mathbb{R}^n$), l'expression locale de P soit un opérateur pseudo-différentiel habituel d'ordre m (au sens de [4] par exemple), tel que pour tout ouvert relativement compact $\omega \subset \Omega$,

$$Pu(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} p(x, \xi) \hat{u}(\xi) d\xi + Ru(x), \quad u \in \mathcal{C}_0^\infty(\omega), \quad x \in \omega,$$

où R est un régularisant analytique (c'est-à-dire son noyau distribution est une fonction analytique sur $\omega \times \omega$). La fonction $p \in \mathcal{C}^\infty(\omega \times \mathbb{R}^n)$ est telle que pour tout compact $K \subset \omega$, il existe les constantes positives A , B et C telles que pour tous multi-indices α et β on ait

$$i) \quad |D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{K, \beta} A^{|\alpha|} \alpha! (1 + |\xi|)^{m - |\beta|}, \quad x \in K, \quad \xi \in \mathbb{R}^n,$$

ou $C_{K, \beta}$ est une constante positive dépendant de K et β ,

$$ii) \quad |D_x^\alpha D_\xi^\beta p(x, \xi)| \leq CA^{|\alpha + \beta|} \alpha! \beta! (1 + |\xi|)^{m - |\beta|}, \quad x \in K, \quad \xi \in \mathbb{R}^n, \quad |\xi| \geq B$$

on aurait pu prendre $|\xi| \geq B|\beta|$.

Alors $P(\mathcal{C}^\infty(\Gamma)) \subset \mathcal{C}^\infty(\Gamma)$ et P admet une extension continue de $\mathcal{D}'(\Gamma)$ à $\mathcal{D}'(\Gamma)$ telle que pour tout $u \in \mathcal{D}'(\Gamma)$ et tout ouvert $U \subset \Gamma$, u analytique sur U entraîne l'analyticité de Pu sur U .

Si P est d'ordre $m \in \mathbb{N}^*$, on peut adapter les raisonnements de Baouendi et Goulaouic pour prouver que pour toute fonction u analytique sur Γ on a les estimations (1). On en déduit que si $m = p/q \in \mathbb{Q}_+^*$, $p, q \in \mathbb{N}^*$, pour toute fonction u analytique sur Γ , il existe $C > 0$, telle que pour tout $k \in \mathbb{N}$,

$$(2) \quad \|P^{qk}u\|_{L^2(\Gamma)} \leq C^{k+1}(k!)^p.$$

Réciproquement, on a le

Théorème. Si P est un opérateur pseudo-différentiel elliptique analytique d'ordre $m = p/q$, $p, q \in \mathbb{N}^*$ et si $u \in \mathcal{C}^\infty(\Gamma)$ vérifie les estimations (2), alors u est analytique.

Démonstration. Suivant une idée de Lions et Magenes [7] on définit une fonction v sur $\Gamma \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ petit, par

$$v(x, t) = \sum_{k=0}^{\infty} \frac{t^{kp}}{(kp)!} (P^{kp} u)(x).$$

On peut considérer par (2) v fonction analytique en $t \in (-\varepsilon, \varepsilon)$, à valeurs dans $L^2(\Gamma)$. Soit (Ω, χ) une carte analytique sur Γ et soient ω_0 et ω deux ouverts relativement compacts tels que $\omega_0 \subset \omega \subset \Omega$. On identifie ω_0, ω, Ω et leurs images par χ . On considère ensuite $\psi_1 \in \mathcal{C}_0^\infty(\omega)$, $\psi_1 = 1$ sur ω_0 , $\psi_2 \in \mathcal{C}_0^\infty((-\varepsilon, \varepsilon))$, $\psi_2 = 1$ sur $\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, $\psi(x, t) = \psi_1(x) \psi_2(t)$, $w = \psi v$.

Soit $Q = \partial_t^p - P^q$; alors $Qv = 0$ et $Qw = Q(\psi - 1)v$ sur $\Gamma \times (-\varepsilon, \varepsilon)$. En coordonnées locales, $Q = \partial_t^p - P_q - R_q$, où P_q est un opérateur pseudo-différentiel elliptique analytique d'ordre p et R_q est régularisant analytique sur ω . En ces coordonnées on a donc

$$(3) \quad \partial_t^p w - P_q w = Q(\psi - 1)v + R_q w$$

et on vérifie facilement que la distribution du côté droit de (3) est partiellement analytique (et donc faiblement partiellement analytique) en x sur $\omega_0 \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ (voir [5] pour les définitions).

Par le Théorème de [5], w aussi sera faiblement partiellement analytique en x sur $\omega_0 \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. D'autre part, w est partiellement analytique en t , donc par la Proposition de [5], w sera analytique sur $\omega_0 \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. Pour $t = 0$ on a que $w = u$ sur ω_0 , donc u est analytique sur ω_0 . q.e.d.

Remarque. Si P est un opérateur différentiel elliptique analytique sur Γ , on retrouve le théorème de Kotake-Narasimhan.

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SUR L'ESTIMATION DE L'ERREUR DANS $L^2(\Omega)$ POUR UN PROBLÈME PARABOLIQUE

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On présente une méthode de résolution approximative du problème parabolique mixte ayant des données nulles à la frontière, en obtenant une estimation de l'erreur dans la norme de l'espace $L^2(\Omega)$. La méthode consiste dans la discrétisation de la variable temps et la résolution des problèmes elliptiques qui en résultent, par la méthode des éléments finis.

1. Introduction

Le but de cet article est de présenter une méthode de résolution approximative du problème mixte pour l'équation parabolique $\partial u / \partial t - Lu = f(x)$, ayant les données nulles à la frontière, et aussi d'obtenir une estimation de l'erreur dans la norme de l'espace $L^2(\Omega)$ en certaines suppositions de régularité pour la condition initiale et pour la frontière du domaine. Le résultat principal de cet ouvrage est le Théorème 4.1, qui établit l'estimation du type $\|u^n - u_{kh}^n\| \leq 0(h + k)$. La méthode consiste dans la discrétisation relativement à la variable temps, d'où il résulte une suite de problèmes elliptiques correspondants aux temps fixés, qui seront résolus d'une manière récursive par la méthode des éléments finis. Le problème qui conduit à la solution approximative peut être résolu à l'aide du calculateur. Dans ce qui suit, on désigne par C une constante positive générique indépendante des h, k, n, g, f .

On considère pour $u = u(x, t)$ le problème de Cauchy-Dirichlet représenté par les conditions suivantes :

$$(1) \quad \begin{cases} \partial u / \partial t - (\operatorname{div}(p(x) \nabla u) - q(x)u) = f(x) & (x, t) \in \Omega \times (0, T) \\ u(x, t) = 0 & (x, t) \in \partial \Omega \times (0, T) \\ u(x, 0) = g(x) & x \in \Omega \end{cases}$$

où Ω est un domaine borné de $\mathbb{R}^m$ ayant la frontière $\partial \Omega$, $p(x) \in C^1(\overline{\Omega})$, $q(x) \in C(\overline{\Omega})$, $p(x) \geq p_0 > 0$ et f, g sont deux fonctions données.

On introduit la suivante classe de fonctions :

$$\mathcal{U} = C([0, T]; L^2(\Omega)) \cap C^1((0, T); L^2(\Omega)) \cap C((0, T); H(\Omega)).$$

Pour $f \in L^2(\Omega)$ et $g \in L^2(\Omega)$, on appelle *solution généralisée* du problème (1), une fonction u qui satisfait les conditions suivantes :

$$(2) \quad \begin{cases} u \in \mathcal{U} \\ (\partial u(t) / \partial t, v(t)) + b(u(t), v(t)) = (f, v(t)) \quad \forall t \in (0, T), \quad \forall v \in \mathcal{U} \\ u(0) = g \end{cases}$$

où $(\cdot, \cdot)$ signifie le produit scalaire de $L^2(\Omega)$, et

$$b(u(t), v(t)) = \int_{\Omega} (p(x) \nabla u \nabla v + q(x) uv) dx.$$

On note par $\{\psi_i\}_{i \in \mathbb{N}}$ les fonctions propres généralisées du problème de Dirichlet se rapportant à l'opérateur $-L$ ($Lu = \operatorname{div} (p(x) \nabla u - q(x) u)$) et par $\{\lambda_i\}_{i \in \mathbb{N}}$ les valeurs propres associées, donc :

$$(3) \quad \begin{cases} \psi_i \in H_0^1(\Omega) & \text{telle que} \\ b(\psi_i, v) = \lambda_i (\psi_i, v), & \forall v \in H_0^1(\Omega), \quad \forall i \in \mathbb{N}. \end{cases}$$

Les valeurs propres λ_i sont des nombres positifs et forment une suite monotone croissante, et les fonctions propres ψ_i forment un système orthonormé et complet dans $L^2(\Omega)$. On en conclut que toute fonction $v \in L^2(\Omega)$ est représentée par

$$v = \sum_{i=1}^{\infty} (v, \psi_i) \psi_i$$

et on a l'égalité :

$$(4) \quad \|v\| = \left(\sum_{i=1}^{\infty} (v, \psi_i)^2 \right)^{1/2}$$

ou $\|\cdot\|$ signifie la norme de $L^2(\Omega)$. On note aussi $\|\cdot\|_j$ la norme de $H^j(\Omega)$ $j = 0, 1, 2$.

De plus, si $\partial\Omega \in C^2$ et $v \in H^2(\Omega) \cap H_0^1(\Omega)$, alors on aboutit à l'inégalité suivante :

$$(5) \quad \sum_{i=1}^{\infty} (v, \psi_i)^2 \lambda_i^2 \leq C \|v\|_2^2$$

(à voir par exemple [2])

La solution généralisée définie par les conditions (2), s'exprime par la série :

$$(6) \quad u(x, t) = \sum_{i=1}^{\infty} (e^{-\lambda_i t} (g, \psi_i) + (1 - e^{-\lambda_i t}) \lambda_i^{-1} (f, \psi_i)) \psi_i$$

2. Le problème semi-discret

La première étape consiste dans la discrétisation du problème (2) se rapportant à la variable t , utilisant un schéma implicite aux différences finies. Dans ce but, pour $N \in \mathbb{N}$ déjà connu, posons $k = T/N$ et $u^n(x) = u(x, nk)$, $n = \overline{0, N}$. Si on remplace la dérivée par rapport à la variable

t , par un quotient aux différences régressives, on définit une solution approximative $u_k^n = u_k^n(x)$, $n = \overline{1, N}$, $u_k^0 = g$, par les conditions suivantes :

$$(7) \quad \begin{cases} u_k^n \in H_0^1(\Omega) \text{ telle que} \\ (u_k^n, v) + kb(u_k^n, v) = (u_k^{n-1}, v) + k(f, v) \quad \forall v \in H_0^1(\Omega). \end{cases}$$

On vérifie aisément que la solution du problème(7) s'exprime par la série :

$$u_k^n = \sum_{i=1}^{\infty} ((1 + k\lambda_i)^{-1} (u_k^{n-1}, \psi_i) + k(1 + k\lambda_i)^{-1} (f, \psi_i)) \psi_i,$$

et d'une manière récursive on obtient

$$(8) \quad u_k^n = \sum_{i=1}^{\infty} ((1 + k\lambda_i)^{-n} (g, \psi_i) + (1 - (1 + k\lambda_i)^{-n}) \lambda_i^{-1} (f, \psi_i)) \psi_i$$

Lemme 2.1. Il existe une constante $C > 0$, telle que

$$(9) \quad |(1 + \tau)^{-n} - e^{-n\tau}| < C \tau, \quad \forall n \in \mathbb{N}$$

pour tout $\tau > 0$.

Théorème 2.1. Si $\partial\Omega \in C^2$ et $g \in H^2(\Omega) \cap H_0^1(\Omega)$, alors il existe une constante $C > 0$, telle que

$$(10) \quad \|u^n - u_k^n\| \leq Ck(\|g\|_2^2 + \|f\|), \quad \forall n = \overline{0, N}$$

Démonstration. Compte tenu des (6), (8), (9), (5) et (4), on obtient successivement :

$$\begin{aligned} \|u^n - u_k^n\| &= \left\| \sum_{i=1}^{\infty} (g, \psi_i) (e^{-\lambda_i n k} - (1 + k\lambda_i)^{-n}) \psi_i + \right. \\ &\quad \left. + \sum_{i=1}^{\infty} (f, \psi_i) \lambda_i^{-1} ((1 + k\lambda_i)^{-n} - e^{-\lambda_i n k}) \psi_i \right\| \leq \\ &\leq \left(\sum_{i=1}^{\infty} (g, \psi_i)^2 |e^{-\lambda_i n k} - (1 + k\lambda_i)^{-n}|^2 \right)^{1/2} + \\ &\quad + \left(\sum_{i=1}^{\infty} (f, \psi_i)^2 \lambda_i^{-2} |e^{-\lambda_i n k} - (1 + k\lambda_i)^{-n}|^2 \right)^{1/2} \leq \\ &\leq Ck \left(\sum_{i=1}^{\infty} (g, \psi_i)^2 \lambda_i^{+2} \right)^{1/2} + Ck \left(\sum_{i=1}^{\infty} (f, \psi_i)^2 \right)^{1/2} \leq \\ &\leq Ck(\|g\|_2 + \|f\|). \end{aligned}$$

3. Le problème discret.

La deuxième étape consiste dans la discrétisation du problème (7) relativement à la variable x , en employant la méthode des éléments finis. Dans ce but on établit d'abord plusieurs données. Considérons

$$a(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$$

définie par

$$a(w, v) = (w, v) + kb(w, v).$$

La forme bilinéaire $a(\cdot, \cdot)$ définit sur $H_0^1(\Omega)$ un produit scalaire $(\cdot, \cdot)_a = a(\cdot, \cdot)$, qui induit sur $H_0^1(\Omega)$ la norme $\|\cdot\|_a : \|v\|_a^2 = (v, v)_a$. Compte tenu de la double inégalité

$$(11) \quad \min(kp_0, 1) \|v\|_1^2 \leq \|v\|_a^2 \leq \max(1, Q, P) \|v\|_1^2$$

où $P = \sup_{x \in \bar{\Omega}} p(x)$, $Q = \sup_{x \in \bar{\Omega}} q(x)$, il résulte que l'espace $H_a = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_a}$ coïncide avec l'espace $H_0^1(\Omega)$. De plus, pour tout $w \in L^2(\Omega)$, on a

$$|(w, v)| \leq \|w\| \|v\| \leq \|w\| \|v\|_a \quad \forall v \in H_a,$$

donc $(w, \cdot)$ est un élément de H_a' , l'espace dual de H_a .

On formule de nouveau les données du problème (7), par les conditions : $u_k^0 = g$ et pour tout $n = \overline{1, N}$

$$(12) \quad \begin{cases} u_k^n \in H_a & \text{telle que} \\ (u_k^n, v)_a = (u_k^{n-1} + kf, v) & \forall v \in H_a. \end{cases}$$

On considère V_h un sous-espace à dimension finie de $H_0^1(\Omega) = H_a$, ayant la propriété suivante :

$$(13) \quad \begin{cases} \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \exists \pi_h v \in V_h & \text{telle que} \\ \|v - \pi_h v\|_j \leq Ch^{2-j} \|v\|_2, & j = 0, 1. \end{cases}$$

La réalisation de V_h et ses propriétés d'interpolation sont obtenues par la méthode des éléments finis (à voir par exemple [1])

C'est maintenant qu'on peut définir une nouvelle solution approximative, par les conditions suivantes : $\bar{u}_{kh}^0 = \pi_h g$ et pour tout $n = \overline{1, N}$

$$(14) \quad \begin{cases} \bar{u}_{kh}^n \in V_h & \text{telle que} \\ (\bar{u}_{kh}^n, v_h)_a = (u_k^{n-1} + kf, v_h) & \forall v_h \in V_h, \end{cases}$$

ces conditions résultant à la suite de discrétisation, relativement à la variable x , de la suite de problèmes donnés par (12).

L'existence et l'unicité de la solution du problème (12) et aussi du problème (14), résultent par le théorème de Riesz dans l'espace de Hilbert $(H_a, (\cdot, \cdot)_a)$ et respectivement $(V_h, (\cdot, \cdot)_a)$.

Théorème 3.1. Si $\partial\Omega \in C^2$ et $g \in H^2(\Omega) \cap H_0^1(\Omega)$, alors on a

$$(15) \quad \|u_k^n - \bar{u}_{kh}^n\|_a^2 \leq Ch^2(h^2 + k) \|g\|_2^2 \quad \forall n = \overline{0, N}.$$

Démonstration. Compte tenu de fait que le produit scalaire $(\cdot, \cdot)_a$ est continu et coercif sur l'espace H_a , si on applique le lemme de Cèa, il résulte :

$$\|u_k^n - \bar{u}_{kh}^n\|_a^2 \leq \inf_{V_h \in V_h} \|u_k^n - v_h\|_a^2 \leq \|u_k^n - \pi_h u_k^n\|_a^2.$$

Vu les conditions de l'énoncé du théorème, sont réalisées les suivantes propriétés de régularité pour la solution u_k^n , $n = \overline{1, N}$.

$$(16) \quad \begin{cases} u_k^n \in H^2(\Omega) \cap H_0^1(\Omega) \\ \|u_k^n\|_2 \leq C \|Lu_k^n\| \end{cases}$$

(à voir par exemple [2])

De (8), (13), (16) il résulte donc :

$$\begin{aligned} \|u_k^n - \bar{u}_{kh}^n\|_a^2 &\leq (1 + kQ) \|u_k^n - \pi_h u_k^n\|^2 + kP \|u_k^n - \pi_h u_k^n\|_1^2 \leq \\ &\leq Ch^2(h^2 + k) \|u_k^n\|_2^2 \\ \|u_k^n\|_2^2 &\leq C \|Lu_k^n\|^2 = C \sum_{i=1}^{\infty} \lambda_i^2 (u_k^n, \psi_i)^2 = \\ &= C \sum_{i=1}^{\infty} (1 + k \lambda_i)^{-2n} (g, \psi_i)^2 \lambda_i^2 \leq \\ &\leq C \sum_{i=1}^{\infty} (g, \psi_i)^2 \lambda_i^2 \leq C \|g\|_2^2 \end{aligned}$$

et par suite de ces deux inégalités, on a la relation (14)

4. L'estimation de l'erreur.

On met en évidence ces problèmes :

$$(17) \quad (u_k^n, v)_a = (u_k^{n-1} + kf, v), \quad \forall v \in H_a$$

$$(18) \quad (\bar{u}_{kh}^n, v_h) = (\bar{u}_{kh}^{n-1} + kf, v_h), \quad \forall v_h \in V_h$$

$$(19) \quad (u_k^n, v_h)_a = (u_{nk}^{n-1} + kf, v_h), \quad \forall v_h \in V_h$$

où $u_k^0 = g$, $\bar{u}_{kh}^0 = \pi_h g$, $u_{kh}^0 = \pi_h g$, $n = \overline{1, N}$.

On remarque aussi que le problème (19) est le seul qui ait une résolution numérique.

Theoreme 4.1. Si $\partial\Omega \in C^2$, $g \in H^2(\Omega) \cap H_0^1(\Omega)$ et $h^2 \leq c_0 k$, alors on a

$$(20) \quad \|u^n - u_{kh}^n\| \leq C(h + k) \|g\|_2 + Ck \|f\|, \quad \forall n = \overline{0, N}.$$

Démonstration. Puisque $V_h \subset H_a$, il résulte des (17) et (18) la relation suivante :

$$(u_h^n - \bar{u}_{kh}^n, v_h)_a = 0 \quad \forall v_h \in V_h.$$

C'est une relation qui nous permet d'établir l'égalité :

$$(21) \quad \|u_k^n - u_{kh}^n\|_a^2 = \|u_k^n - \bar{u}_{kh}^n\|_a^2 + \|\bar{u}_{kh}^n - u_{kh}^n\|_a^2$$

Des (18) et (19), on a

$$\begin{aligned} \|\bar{u}_{kh}^n - u_{kh}^n\|_a^2 &= (\bar{u}_{kh}^n - u_{kh}^n, u_k^{n-1} - u_{kn}^{n-1}) \leq \\ &\leq \|\bar{u}_{kh}^n - u_{kh}^n\|_a \cdot \|u_k^{n-1} - u_{kn}^{n-1}\|_a. \end{aligned}$$

Compte tenu du Théorème 3.1, on obtient

$$\|u_k^n - u_{kh}^n\|_a^2 \leq \|u_k^{n-1} - u_{kh}^{n-1}\|_a^2 + Ch^2(h^2 + k) \|g\|_2^2.$$

Récursivement, on en obtient

$$\begin{aligned} \|u_k^n - u_{kh}^n\|_a^2 &\leq nCh^2(h^2 + k)_2^2 \|g\|_2^2 + \|g - \pi_h g\|_a^2 \leq \\ &\leq (n + 1)Ch^2(h^2 + k) \|g\|_2^2 \leq (c_0 + 1)(n + 1)kCh^2 \|g\|_2^2 \leq \\ &\leq Tch^2 \|g\|_2 = Ch^2 \|g\|_2^2. \end{aligned}$$

Finalement on aura donc

$$\|u_k^n - u_{kh}^n\| \leq Ch \|g\|_2.$$

En vertu du Théorème 2.1, on obtient l'estimation suivante :

$$\|u^n - u_{kh}^n\| \leq C(h + k) \|g\|_2 + Ck \|f\|.$$

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NUMERICAL RESULTS IN DEGENERATING A COMPACT HYPERELLIPTIC RIEMANN SURFACE OF GENUS 4*

AARON LEBOWITZ AND HAROLD SHULMAN

Period matrices are computed for a number of degenerations of a compact hyperelliptic Riemann surface of genus 4.

1. Introduction

In an earlier paper [1], a number of degenerations of a compact hyperelliptic Riemann surface of genus 4 were studied, using theta function techniques. In that paper formulas for the branch points on the surface were obtained as quotients of products of certain Riemann theta constants. The degeneration was obtained by studying the effect on the branch point formulas of degenerating the period matrix.

This paper deals with the dual problem. Namely, can one say something about the effect on the period matrix of coalescing the branch points. We refer the reader to [1, Figure 1], to see which are the γ and δ cycles on the degenerating Riemann surface.

A computer technique was applied to obtain the normalized period matrix of a genus 4 hyperelliptic Riemann surface, and the computation was repeated successively for numerous positions of the branch points as the points coalesced. Thus, for each degeneration, a sequence of period matrices was obtained as the branch points were moved incrementally toward the limiting positions. The present results deal with the very special case where all finite branch points, λ_i , are real, initially placed at the points $-2, -1, 0, 1, 2, 3, 4, 5, 6$. Then various degenerations are studied.

2. The integration procedure

All integrations were taken around cycles which were started at the same point, namely,

$$p = \frac{\sum_{i=1}^q R_2(\lambda_i)}{q} + i (\max_{1 \leq i \leq q} \text{Im}(\lambda_i) + 1).$$

A γ cycle enclosing λ_i and λ_{i+1} followed the contour shown in Figure 1. TQ and RS are horizontal; QR is vertical and passes through the midpoint of the line segment $\overline{\lambda_{i-1} \lambda_i}$. Similarly for ST (through the line segment $\overline{\lambda_{i+1} \lambda_{i+2}}$). Figure 2 indicates the contour of a δ cycle.

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The numerical procedure used was Simpson's rule on each straight line segment. A number $H > 0$ was chosen and then N , the number of

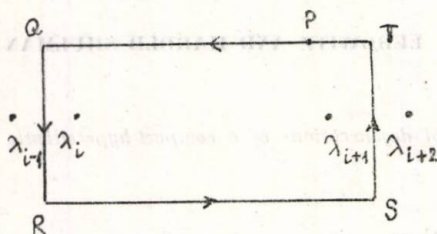


Fig. 1

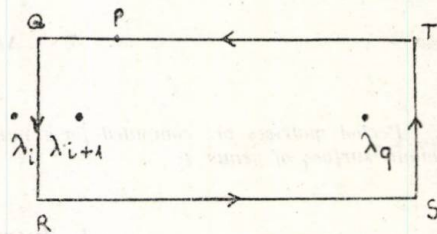


Fig. 2

intervals was determined by the formula

$$N = \left\lceil \frac{\text{length of line segment}}{H} \right\rceil + 1,$$

finally the step size in the integration procedure was chosen as

$$h = \frac{\text{length of line segment}}{2N}.$$

If a line segment (QR or ST) came „very close” to a λ_i , that line-segment was split into three parts, and, on the central part a much smaller value of H was introduced. Thus many more points were used when the integration contour was near a singular point.

The integrand involves the square root of a complex number. The initial branch of that square root was computed at P (the FORTRAN routine always yields the square root with positive real part) and at each succeeding point the branch was chosen which was closest to the value at the previous point.

If $|\lambda_{i-1} - \lambda_i|$ was very small, this method of choosing the proper branch of the square root may not have worked well, particularly when the path of integration passed close to λ_i ; perhaps a much smaller step size or double precision computation (or both) may produce more reliable results. Another explanation for poor results for $|\lambda_{i-1} - \lambda_i|$ very small, may be that the matrix for the γ cycles, the A -matrix, is nearly singular (ill conditioned in numerical analysis parlance) and the numerical calculation of its inverse may not be very accurate even though it is only a 4×4 matrix.

No actual calculation was carried out of an inverse, as it was unnecessary. We simply applied elementary transformations to the A -matrix until it was reduced to I , the identity matrix, and we applied the same transformations to the matrix for the δ -cycles, the B -matrix. This saves time and reduces round off error. (Calculating A first and then $A^{-1} B$ would introduce more operations and hence more round off errors).

The computations were made on the CUNY IBM 370/168 in single precision complex arithmetic.

3. Results

In this section we will list some of the results which we have obtained.

Recall that all finite branch points are taken real and placed at $-2, -1, 0, 1, 2, 3, 4, 5, 6$ as a starting position.

Note the important observation that all periods, Π_{ij} , are therefore pure imaginary. What we are giving is actually the approximate absolute value of each period, which we have denoted as the usual, Π_{ij} .

Case 1. Allow $1 \rightarrow 0$. Π_{ij} remained fairly constant except for Π_{22} which grew from approximately 1.3 to 2175 by the time 1 had been replaced by 0.4882×10^{-6} . Π_{22} continued to increase very rapidly but at this point we began to lose symmetry of the matrix.

$$\Pi_{11} \sim 1.511, \Pi_{21} \sim 0.914, \Pi_{31} \sim 0.519, \Pi_{41} \sim 0.305,$$

$$\Pi_{32} \sim 0.607, \Pi_{42} \sim 0.347, \Pi_{33} \sim 1.252, \Pi_{43} \sim 0.497, \Pi_{44} \sim 0.995.$$

Case 2. Allow $1 \rightarrow 0$ and $4 \rightarrow 3$. This degeneration exhibited the following interesting behavior. Π_{11} remained fairly constant at 1.47, $\Pi_{21} \sim 0.868$, Π_{22} grew as in case 1, and rows three and four tended to one another, i.e., $\Pi_{3i} \sim \Pi_{4i}$ for $i = 1, 2, 3, 4$. This was computed while 1 was being replaced by 0.762×10^{-5} and 4 by 3.00001.

Case 3. In this case we allowed both 1 and 2 to tend toward 0. The following fascinating result was obtained. Nothing became infinite, but $\Pi_{2i} \cong \Pi_{3i}$ for $i = 1, 3, 4$ and $\Pi_{32} \cong \Pi_{22} - 1$. That is, the second and third rows were essentially identical except for the second elements which differed by 1. This was obtained while 1 was being replaced by 0.3051×10^{-4} and 2 by 0.6103×10^{-4} .

While we have some further results we will give no more cases at this time since we lose symmetry before we are able to say anything definitive or nice. We have alluded, in section 2, above, to the difficulties which must be overcome.

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ON THE DUAL PREDICTABLE PROJECTION OF A PROCESS WITH INTEGRABLE VARIATION

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The paper deals with a certain representation for a process with integrable variation which enable us to find out the structure of the dual predictable projection of the process and also to do some remarks about the structure of a martingale with integrable variation.

Let $\{Q, F, P\}$ be a complete probability space and let $(F_t)_{t \in R_+}$ be an increasing family of sub- σ -fields of F , such that for any $t \in R_+$, we have $F_t = \bigcap_{s>t} F_s$ and F_0 contains all sets of F measure zero.

We shall use freely notations, terminology and results from [2].

Let A be a real, adapted, right continuous process and consider the variation of A , whose state at time t is the random variable $\omega \rightarrow$

$$\rightarrow \int_0^t |dA_s(\omega)|, \text{ where}$$

$$\int_0^t |dA_s(\omega)| = |A_0(\omega)| + \sup_n \sum_{i=0}^{2^n-1} |A_{t_{i+1}^{(n)}}(\omega) - A_{t_i^{(n)}}(\omega)| t_i^{(n)} = \frac{it}{2^n}$$

$$i = 0, 1, 2, \dots, 2^n$$

If the total variation of A , i.e. the random variable

$$\int_0^\infty |dA_s(\omega)| = \lim_{t \rightarrow \infty} \int_0^t |dA_s(\omega)|$$

is integrable, then we say that the process A is with integrable variation.

We denote by $\int |dA|$ the variation of A .

Let A be a process with integrable variation. Then A can be represented in an unique manner in the following way :

$$(1) \quad A = \tilde{A} + \overset{c}{A}$$

where $\tilde{A}$ is a predictable process with integrable variation („predictable” means measurable with respect to the σ -field of predictable sets, which is the σ -field generated on $\Omega \times R_+$ by all real adapted continuous processes) and $\overset{c}{A}$ is a martingale vanishing in zero. $\tilde{A}$ is called the dual predictable

projection of A or the compensator of A and $\tilde{A}$ is called A -compensated. The process $\tilde{A}$ appears in a natural way passing through Radon-Nikodym theorem. Excepting for the right continuity, the properties of the trajectories of $\tilde{A}$ are not at all obvious and an aim of this paper is to find out them, more precisely to establish connections between them and the trajectories of A .

Let us consider now the representation

$$(2) \quad A = A^c + A^d$$

where A^c is the continuous part of A and A^d is its purely discontinuous part. The process A^d is obtained in the following way; we consider a countable sequence of stopping times (T_n) , such that $T_0 = 0$, $T_n > 0$ a.e., for any $n = 1, 2, 3, \dots$ and

$$(3) \quad \{(t, \omega) \mid \Delta A_t(\omega) \neq 0\} \subset \bigcup_{n=0}^{\infty} [T_n]$$

where $\Delta A_t(\omega) = A_t(\omega) - A_{t-}(\omega)$, $A_0(\omega) = 0$ and $[T_n]$ denotes the graph of T_n . One may suppose that the union in the right side of (3) is a disjoint union and that each T_n is either predictable or totally inaccessible stopping time. Then

$$(4) \quad A^d = \Sigma A^{(n)} \text{ where } A^{(n)} = \Delta A_{T_n} I_{[T_n, \infty)}$$

and $[T_n, \infty) = \{(t, \omega) \mid T_n(\omega) \leq t\}$, $I_{[T_n, \infty)}$ being its indicator function. We have also

$$(5) \quad \int |dA| = \int |dA^c| + \int |dA^d|$$

and this is the decomposition (2) for $\int |dA|$.

The tools we need further are the following two lemmas:

Lemma 1. Let $A = \Phi I_{[T, \infty)}$, where T denotes a stopping time, Φ is an integrable random variable measurable with respect to F_T and such that $\{T = \infty\} \subset \{\Phi = 0\}$, a. s. and $I_{[T, \infty)}$ denotes the indicator of the stochastic interval $[T, \infty)$. Then the following implications hold:

a) If T is totally inaccessible, then $\tilde{A}$ is continuous.

b) If T is predictable, then $A = E(\Phi | F_{T-}) I_{[T, \infty)}$

Proof. The point a) is a result from [1] (VIII, T. 31). However, in order that this note be selfcontained we sketch the proof. The Doléans measure,

$$PA(x) = \iint x_t dA_t dP$$

where x is any real bounded measurable process, is considered. The measure $P\tilde{A}$ coincides with PA on all predictable sets, hence it cannot charge the graph of any predictable stopping time. But the jumps of $\tilde{A}$ are supported only by a countable number of graphs of predictable stopping times, hence $\tilde{A}$ is continuous.

b) If T is predictable, then the equality

$$A = fI_{[T, \infty)} + gI_{[T, \infty)}$$

where $f = E(\Phi | F_{T-})$ and $g = \Phi - E(\Phi | F_{T-})$ is the decomposition (1) for A . Indeed, $fI_{[T, \infty)}$ is a predictable process with integrable variation and $gI_{[T, \infty)}$ is a martingale (since $\int gh dP = 0$ for any bounded F_{T-} -measurable random variable h) which vanishes in zero.

Remark. If A is that from the point a), then

$$(6) \quad \overset{c}{A} = (-\tilde{A}) + A$$

represents the decomposition (2) for the process $\overset{c}{A}$. We have also, using (5)

$$(7) \quad \int |d\overset{c}{A}| = \int |d\tilde{A}| + \int |dA|$$

when $\Phi \geq 0$, the relation (6) represents also the Jordan decomposition of $\overset{c}{A}$.

Lemma 2. If A is a process with integrable variation, then

$$(8) \quad \int |d\tilde{A}| \leq \overline{\int |dA|}$$

Proof. Let us consider the Jordan decomposition of A and $\tilde{A}$

$$A = A^+ - A^- \quad \tilde{A} = (\tilde{A})^+ - (\tilde{A})^-$$

where A^+ , A^- , $(\tilde{A})^+$, $(\tilde{A})^-$ are nonnegative increasing processes, integrable at $+\infty$, which satisfy

$$\int |dA| = A^+ + A^- \quad \int |d\tilde{A}| = (\tilde{A})^+ + (\tilde{A})^-$$

Since the map $A \rightarrow \tilde{A}$ is linear, we have $\tilde{A} = \tilde{A}^+ - \tilde{A}^-$, hence from the properties of the Jordan decomposition it results that $(\tilde{A})^+ \leq (\tilde{A}^+)$ and $(\tilde{A})^- \leq (\tilde{A}^-)$. But

$$\overline{\int |dA|} = (\tilde{A}^+) + (\tilde{A}^-)$$

and we get

$$\int |d\tilde{A}| \leq \int \overline{|dA|}$$

Remark. a) If A is that from Lemma 1 b), then we have

$$\int |d\tilde{A}| = |E(\Phi | F_{T-})| I_{[T, \infty)}$$

and

$$\int \overline{|dA|} = E(|\Phi| | F_{T-}) I_{[T, \infty)}$$

hence in (7) we can't expect, in general, more than inequality.

b) From (8) and the relations

$$(9) \quad \int |dA| = \int \overline{|dA|} + \int \overline{\overline{|dA|}^c}, \quad \int |dA| \leq \int |d\tilde{A}| + \int |d\tilde{A}^c|,$$

we deduce that

$$\int \overline{\overline{|dA|}^c} \leq \int |d\tilde{A}^c|$$

When A is the process from Lemma 1 a), using (7) and (9) we deduce that

$$\int |d\tilde{A}^c| - \int \overline{\overline{|dA|}^c} = \int |d\tilde{A}| + \int \overline{|dA|};$$

moreover, if the function $\Phi \geq 0$, then this equality becomes

$$\int |d\tilde{A}^c| - \int \overline{\overline{|dA|}^c} = 2\tilde{A}$$

Theorem 3. Let A be a process with integrable variation and let (R_n) and (S_n) be the sequences of stopping times with disjoint graphs, the first of totally inaccessible stopping times, the second of predictable stopping times, such that

$$\{(t, \omega) | \Delta A_t(\omega) \neq 0\} \subset \bigcup_n [R_n] \cup \bigcup_n [S_n]$$

Then, the dual predictable projection of A and A -compensated have the following expressions:

$$(10) \quad \tilde{A} = A^c + \sum_n \tilde{B}^{(n)} + \sum_n \tilde{C}^{(n)}$$

$$(11) \quad \overset{c}{A} = \sum_n \overset{c}{B}^{(n)} + \sum_n \overset{c}{C}^{(n)}$$

where all processes in the right side are with integrable variation and all series are absolutely convergent and uniformly convergent with respect to t on almost all trajectories. In the above expressions, for any natural n , we have

$$B^{(n)} = \Delta A_{R_n} I_{[R_n, \infty)}$$

$\tilde{B}^{(n)}$ is continuous,

$$\tilde{C}^{(n)} = E(\Delta A_{S_n} | F_{S_n-}) I_{[S_n, \infty)}$$

$\overset{c}{B}^{(n)}$ and $\overset{c}{C}^{(n)}$ are vanishing in zero martingales, continuous excepting the graph of R_n and S_n , respectively. Moreover,

$$\Delta \overset{c}{B}_{R_n}^{(n)} = \Delta A_{R_n} \text{ and } \overset{c}{C}^{(n)} = [A_{S_n} - E(A_{S_n} | F_{S_n-})] T_{S_n}$$

Proof. Let us consider the expressions (4) for the discontinuous part of A . The sum in the right side is absolutely convergent and uniformly convergent in t on almost all trajectories. Indeed, for any t , we have

$$\sum_n |A^{(n)}(t, \omega)| \leq \sum_n |\Delta A_{T_n}(\omega)| = \int_0^\infty |dA^d(\omega)|;$$

using (5), this last term is an integrable random variable, hence it is finite almost everywhere.

Let now consider, for any $n \in N$, the representation

$$A^{(n)} = \tilde{A}^{(n)} + \overset{c}{A}^{(n)}$$

All we have said about the sum $\Sigma A^{(n)}$, is now true for $\Sigma \tilde{A}^{(n)}$ and consequently for $\Sigma \overset{c}{A}^{(n)}$ also. Indeed,

$$\begin{aligned} \sum_n |\tilde{A}^{(n)}(t, \omega)| &\leq \sum_n \int_0^\infty |d\tilde{A}^{(n)}(\omega)| \leq \sum_n \left(\int_0^\infty |dA^{(n)}(\omega)| \right)_\infty = \\ &= \left(\sum_n \int_0^\infty |dA^{(n)}(\omega)| \right)_\infty = \left(\int_0^\infty |dA^d(\omega)| \right)_\infty \end{aligned}$$

the last term is an integrable random variable, since from (5) we deduce that

$$\int |dA| = \int |dA^c| + \int |dA^d|$$

hence

$$\left(\int |dA^d| \right)_\infty \leq \left(\int |dA| \right)_\infty$$

We have now

$$(12) \quad A = A^c + \sum_n \tilde{A}^{(n)} + \sum_n \overset{c}{A}^{(n)}$$

Let us associate those $A^{(n)}$ for which T_n is totally inaccessible stopping time and redenote them by $(B^{(n)})$. We can suppose that (R_n) is the sequence of all totally inaccessible stopping times from (T_n) . We do the same thing with those $A^{(n)}$ corresponding to a predictable stopping time T_n : we redenote them by $(C^{(n)})$ and assume (S_n) are all the predictable stopping times from (T_n) . Then (12) may be written as follows:

$$(13) \quad A = A^c + \sum_n \tilde{B}^{(n)} + \sum_n \tilde{C}^{(n)} + \sum_n \overset{c}{B}^{(n)} + \sum_n \overset{c}{C}^{(n)}$$

Comparing (13) with (1) and using the unicity of (1) we get

$$\tilde{A} = A^c + \sum_n \tilde{B}^{(n)} + \sum_n \tilde{C}^{(n)}$$

$$\overset{c}{A} = \sum_n \overset{c}{B}^{(n)} + \sum_n \overset{c}{C}^{(n)}$$

The properties of the terms in the right side of this equalities are immediate, applying Lemma 1 b) to each $\tilde{C}^{(n)}$ and the proof is complete.

Using the notations of the above theorem, the following corollary is immediate.

Corollary 4. Let A be a process with integrable variation. Then we have

$$(\tilde{A})^c = A^c + \sum_n \tilde{B}^{(n)}, \quad (\tilde{A})^d = \sum_n \tilde{C}^{(n)} = \sum_n E(\Delta A_{S_n} | \mathcal{F}_{S_n-}) I_{[S_n, \infty)}$$

$$(\overset{c}{A})^c = - \sum_n \tilde{B}^{(n)}, \quad (\overset{c}{A})^d = \sum_n B^{(n)} + \sum_n \overset{c}{C}^{(n)}, \quad (\tilde{A})^c = A^c$$

$$\overset{c}{\hat{A}}^c = 0, \quad (\tilde{A})^d = \sum_n \tilde{B}^{(n)} + \sum_n \tilde{C}^{(n)}$$

(and this is the decomposition (2) for $(\tilde{A}^d)$),

$$(\tilde{A}^d)^c = \tilde{A}, \quad (\tilde{A}^d)^c = -(\tilde{A})^c,$$

and so on.

Remark. Finally, we want to do some comments about the equality (11). It is well known that any martingale with integrable variation which vanishes in zero is equal to the compensated sum of its jumps ([2] II, 9), i.e. is of the form $\tilde{A}$, where A is the purely discontinuous part of the martingale (considered as process with integrable variation). Now, (11) asserts that in this representation one can commute the operator „c” from above of A with the sum. The equality (11) is identical in form with the representation giving the structure of a square integrable martingale which is purely discontinuous ([2], II, 11), but there the meaning of the sum was another, it was convergence in $L^2(F)$. In this way (11) is the same as the representation of a purely discontinuous square integrable martingale but with another meaning of the sum, namely pointwise sum.

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SUFFICIENT OPTIMALITY CONDITIONS IN CONTROL THEORY

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Fixed time-interval optimal control problems defined by differential inclusions or parametrized differential equations are considered and sufficient optimality conditions that are shown to generalize those of R. T. Rockafellar ([11]) in the convex case and those of V. Blagodatskih ([2]) in the non-convex case are obtained.

The well known Pontryagin's Maximum Principle is a necessary optimality condition under rather restrictive hypotheses on the functions and the sets that define the optimal control problem, hence it cannot be applied to some very simple concrete problems where, for instance, „unorthodox” control and phase space constraints occur. Moreover, very simple examples show that it is far from being also a sufficient optimality condition ([12]).

These are cases in which sufficient optimality conditions are needed.

For the so called „convex problems”, very strong sufficient optimality conditions (that are proved to be also necessary under some additional hypotheses) were given by R. T. Rockafellar ([11]) in the form of a generalized Hamiltonian inclusion; for non-convex problems sufficient optimality conditions were given by V. I. Blagodatskih ([2]) in terms of the support function of the problem.

Another type of sufficient optimality conditions, in terms of the Hamilton-Jacobi equation, were given by R. Klötzler ([9]) for a very general optimal control problem and they contain as particular cases Rockafellar's sufficient conditions.

In what follows we consider first a fixed time-interval optimal control problem defined by a differential inclusion and we prove three subsequent sufficient optimality conditions, the last one (Proposition 3) being equivalent, in the convex case, with Rockafellar's sufficient optimality condition; the condition contained in Proposition 2 (which cannot be derived from that in [9]) is shown to be stronger than the condition in [2] for the fixed time-interval problem.

In the last part of the paper we consider the „standard” optimal control problem defined by a parametrized differential equation instead of a differential inclusion.

We denote by R^n the n -dimensional Euclidian space, by $\mathcal{P}(R^n)$ the family of all nonempty subsets of R^n and by $\langle ., . \rangle$ the scalar product in R^n ; if $[t_0, t_1] \subset R$ is an interval then $AC([t_0, t_1]: R^n)$ denotes the space of absolutely continuous mappings from $[t_0, t_1]$ to R^n .

The control problem we are studying is defined by an interval $[t_0, t_1] \subset R$ and by two multifunctions: $F(.): [t_0, t_1] \times R^n \rightarrow \mathcal{P}(R^n)$ $X(.): [t_0, t_1] \rightarrow \mathcal{P}(R^n)$ as follows: the set $\mathcal{A}$ of the mappings $x(.) \in$

$\in AC([t_0, t_1]; R^n)$ that satisfy the conditions :

$$\frac{dx}{dt}(t) \in F(t, x(t)) \text{ a.e. on } [t_0, t_1] \quad (1)$$

$$x(t) \in X(t) \text{ for any } t \in [t_0, t_1] \quad (2)$$

is said to be the set of admissible trajectories of the problem, and for any $t \in [t_0, t_1]$, the set $\mathcal{A}(t)$ defined by :

$$\mathcal{A}(t) = \{x(t); x(\cdot) \in \mathcal{A}\} \quad (3)$$

is said to be the reachable set at the time t .

We assume that the set $\mathcal{A}$ of the admissible trajectories is not empty; for conditions on $F(\dots)$ that ensure the existence of solutions of the differential inclusion (1) we refer to [8].

The optimality criterion of the problem is defined by two lower semi-continuous functions: $g(\cdot, \cdot): X(t_0) \times X(t_1) \rightarrow R$, $f_0(\cdot, \cdot, \cdot): [t_0, t_1] \times R^n \times R^n \rightarrow R$, the last being also a „normal integrand” ([11]) in the sense that it is measurable with respect to the σ -algebra generated by the products of Lebesgue sets in $[t_0, t_1]$ and Borel sets in $R^n \times R^n$; this property implies, in particular, that for any mapping $x(\cdot) \in \mathcal{A}$, the function $t \mapsto f(t, \dot{x}(t), x(t))$ is Lebesgue measurable.

The function $g(\cdot, \cdot)$ and $f_0(\cdot, \cdot, \cdot)$ define the functional to be minimized, $I(\cdot): \mathcal{A} \rightarrow R$ as follows :

$$I(x(\cdot)) = g(x(t_0), x(t_1)) + \int_{t_0}^{t_1} f_0(t, x(t), \dot{x}(t)) dt \quad (4)$$

An admissible trajectory $\tilde{x}(\cdot) \in \mathcal{A}$ is said to be optimal if :

$$I(\tilde{x}(\cdot)) = \min \{I(x(\cdot)); x(\cdot) \in \mathcal{A}\} \quad (5)$$

An essential role in the theory of (necessary or sufficient) optimality conditions is played by the function: $\mathcal{H}: [t_0, t_1] \times R^{3n} \rightarrow R$ defined by :

$$\mathcal{H}(t, x, p, v) = \langle p, v \rangle - f_0(t, x, v), \quad t \in [t_0, t_1], \quad p, x, v \in R^n \quad (6)$$

which is usually called the Hamiltonian of the problem but in [9] is called the „Pontryagin function” and in [6] — the „pseudo-Hamiltonian”.

The sufficient optimality conditions in the sequel will be expressed in terms of the function $H: [t_0, t_1] \times R^{2n} \rightarrow R$ defined by :

$$H(t, x, p) = \sup \{ \mathcal{H}(t, x, p, v); v \in F(t, x) \} \quad (7)$$

which is called „Hamiltonian” in [9] and [11] and „true Hamiltonian” in [6]. It may be called also the „support function” of the optimal control problem since $H(t, x, p)$ is indeed the value at $(p, -1)$ of the support function of the „extended vectogram” of the problem $\bar{F}(t, x) = F(t, x) \times \{f_0(t, x, \dot{x})\}$.

We assume that for every $t \in [t_0, t_1]$ the set $Y(t)$ of the points $(x, p) \in R^n \times R^n$ for which $H(t, x, p)$ is finite is not empty (if $\mathcal{H}$ is a continuous function and $F(\cdot, \cdot)$ is compact-valued then obviously $Y(t) = R^{2n}$ for any $t \in [t_0, t_1]$) and we assume that for any $(x, p) \in Y(t)$ the set of vectors at which the supremum in (7) is attained:

$$V(t, x, p) = \{v \in F(t, x); \mathcal{H}(t, x, p) = H(t, x, p, v)\} \quad (8)$$

is not empty for every $t \in [t_0, t_1]$.

We assume also that the set $\mathcal{B}$ of the pairs of mappings $(x(\cdot), p(\cdot)) \in \mathcal{A} \times AC([t_0, t_1]; R^n)$ for which $(x(t), p(t)) \in Y(t)$ for every $t \in [t_0, t_1]$ is not empty and that the support function $H(\cdot, \cdot, \cdot)$ is a normal integrand; these conditions are satisfied in the optimal control problems in the literature and it is very likely that if they are not satisfied then the problem (1) — (5) does not have an optimal trajectory.

We consider now the functional $I_1(\cdot, \cdot): \mathcal{A} \times AC([t_0, t_1]; R^n) \rightarrow [-\infty, +\infty]$ defined by:

$$I_1(x(\cdot), p(\cdot)) = g^1(x(t_0), x(t_1), p(t_0), p(t_1)) + \int_{t_0}^{t_1} f_0^1(t, x(t), p(t), \dot{p}(t)) dt \quad (9)$$

if $(x(\cdot), p(\cdot)) \in \mathcal{B}$ and $I_1(x(\cdot), p(\cdot)) = -\infty$ otherwise, where the functions g^1 and f_0^1 are defined by:

$$g^1(x_0, x_1, p_0, p_1) = \langle p_0, x_0 \rangle - \langle p_1, x_1 \rangle - g(x_0, x_1) \quad (10)$$

$$f_0^1(t, x, p, q) = \langle q, x \rangle + H(t, x, p) \quad (11)$$

We shall prove first the following general sufficient optimality condition which, in turn, may generate some other, easier verifiable sufficient conditions.

Proposition 1. Let $\tilde{x}(\cdot) \in \mathcal{A}$ be an admissible trajectory of the control problem (1) — (2) such that there exists a mapping $\tilde{p}(\cdot) \in AC([t_0, t_1]; R^n)$ satisfying the following conditions:

$$\frac{d\tilde{x}}{dt}(t) \in V(t, \tilde{x}(t), \tilde{p}(t)) \text{ a.e. on } [t_0, t_1] \quad (12)$$

$$I_1(\tilde{x}(\cdot), \tilde{p}(\cdot)) = \sup \{I_1(\tilde{x}(\cdot), \tilde{p}(\cdot)); x(\cdot) \in \mathcal{A}\} \quad (13)$$

Then $\tilde{x}(\cdot)$ is an optimal trajectory of the problem (1) — (5) and the minimal value of the functional $I(\cdot)$ is given by:

$$I(\tilde{x}(\cdot)) = -I_1(\tilde{x}(\cdot), \tilde{p}(\cdot)). \quad (14)$$

Proof. Obviously the statement to be proved in Proposition is a corollary of the following one: for any admissible trajectory $x(\cdot) \in \mathcal{A}$ we have:

$$I(x(\cdot)) \geq -I_1(x(\cdot), \tilde{p}(\cdot)) \quad (15)$$

and the equality holds if and only if (12) is satisfied.

For any $x(\cdot) \in \mathcal{A}$ we have, obviously: $I(x(\cdot)) = g(x(t_0), x(t_1)) +$

$$\begin{aligned} & + \int_{t_0}^{t_1} f_0(t, x(t), \dot{x}(t)) dt = g(x(t_0), x(t_1)) + \int_{t_0}^{t_1} \left[\frac{d}{dt} \langle \tilde{p}(t), x(t) \rangle - \right. \\ & \left. - \langle \dot{\tilde{p}}(t), x(t) \rangle - \langle \tilde{p}(t), \dot{x}(t) \rangle + f_0(t, x(t), \dot{x}(t)) \right] dt = \\ & = -g^1(x(t_0), x(t_1), \tilde{p}(t_0), \tilde{p}(t_1)) - \int_{t_0}^{t_1} \left[\langle \dot{\tilde{p}}(t), x(t) \rangle + \mathcal{H}(t, x(t), \tilde{p}(t), \right. \\ & \left. \dot{x}(t)) \right] dt \geq -g^1(x(t_0), x(t_1), \tilde{p}(t_0), \tilde{p}(t_1)) - \\ & - \int_{t_0}^{t_1} \left[\langle \dot{\tilde{p}}(t), x(t) \rangle + H(t, x(t), \tilde{p}(t)) \right] dt = -I_1(x(\cdot), \tilde{p}(\cdot)) \end{aligned}$$

(since, according to (7), $\mathcal{H}(t, x(t), p(t), x(t)) \leq H(t, x(t), p(t))$ a.e. for $t \in [t_0, t_1]$) hence (15) is proved.

Since in the case $\tilde{x}(\cdot) \in \mathcal{A}$ satisfies (12) one has $H(t, \tilde{x}(t), \tilde{p}(t)) = \mathcal{H}(t, \tilde{x}(t), \tilde{p}(t), \dot{\tilde{x}}(t))$, the same computations as above prove (14) and Proposition 1 is completely proved.

Remark 2. From (7), (8) and (15) it follows that condition (21) is equivalent with the following one:

$$\inf \{ I_1(\tilde{x}(\cdot), p(\cdot)); p(\cdot) \in AC([t_0, t_1]: R^n) \} = I_1(\tilde{x}(\cdot), \tilde{p}(\cdot)) \quad (16)$$

hence the sufficient optimality conditions in Proposition 1 may alternatively be expressed as follows:

$$\sup_{x(\cdot) \in \mathcal{A}} \inf \{ I_1(x(\cdot), p(\cdot)); p(\cdot) \in AC([t_0, t_1]: R^n) \} = I_1(\tilde{x}(\cdot), \tilde{p}(\cdot)) \quad (17)$$

Remark 3. We note that property (12) coincides with the „maximum property” in the well known Pontryagin's Maximum Principle ([1]) or its generalizations in [5] and elsewhere.

Moreover, it is known that under fairly general hypotheses ([7]) on the multifunction $F(\cdot, \cdot)$ and on the integrand $f_0(\cdot, \cdot, \cdot)$, the support

function $H(\cdot, \cdot, \cdot)$ is locally Lipschitz in the variables (x, p) and in this case, if the terminal payoff $g(\cdot, \cdot)$ is assumed also locally Lipschitz, then the „Euler-Lagrange differential inclusion” in [3] — a necessary optimality condition for $I_1(\cdot, \tilde{p}(\cdot))$ under suitable hypotheses — becomes:

$0 \in \tilde{p}(t) + \partial_x H(t, \tilde{x}(t), \tilde{p}(t))$ and this, in turn, is a necessary optimality condition ([5]) for $I(\cdot)$ ($\partial_x H(t, x, p)$ denotes Clarke's generalized gradient of $H(t, \cdot, p)$). It follows that the necessary optimality conditions of the functionals $I(\cdot)$ and $I_1(\cdot, \tilde{p}(\cdot))$ are very closely related and it seems a very interesting open problem to find conditions under which (12) and (13) are also necessary optimality conditions.

Remark 4. It is obvious that for a fixed mapping $\tilde{p}(\cdot) \in AC([t_0, t_1] : R^n)$ any sufficient optimality condition for maximizing the functional $I_1(\cdot, \tilde{p}(\cdot))$ subject to (12) is also a sufficient optimality condition for the problem (1) — (5).

The advantage of the problem of maximizing the functional $I_1(\cdot, \tilde{p}(\cdot))$ is that its integrand, $f_0^1(\cdot, \cdot, \cdot, \cdot)$ does not depend on $\dot{x}(t)$ (its disadvantage being the fact that the integrand $f_0^1(\cdot, \cdot, \tilde{p}(\cdot), \dot{\tilde{p}}(\cdot))$ is not, generally, a smooth function with respect to the x -variable).

The simplest way to obtain sufficient optimality conditions for the problem of maximizing the functional $I_1(\cdot, \tilde{p}(\cdot))$ on $\mathcal{A}$ is to maximize separately the „terminal payoff”, $g^1(\cdot, \cdot, \tilde{p}(t_0), p(t_1))$ and the „integral”

one, $x(\cdot) \mapsto \int_{t_0}^{t_1} f_0^1(t, x(t), \tilde{p}(t), \dot{\tilde{p}}(t)) dt$. It seems likely that under reasonable

hypotheses, in the absence of the derivative $\dot{x}(t)$ in the integrand f_0^1 , this could be also a necessary optimality condition for the functional $I_1(\cdot, \tilde{p}(\cdot))$. Furthermore, as it was proved by R. T. Rockafellar ([11]), one has;

$$\begin{aligned} \sup \left\{ \int_{t_0}^{t_1} f_0^1(t, x(t), \tilde{p}(t), \dot{\tilde{p}}(t)) dt ; x(\cdot) \in \mathcal{A} \right\} = \\ = \int_{t_0}^{t_1} \sup \{ f_0^1(t, \tilde{p}x, \dot{\tilde{p}}(t), (t)) ; x \in \mathcal{A}(t) \} dt \text{ if } (t, x) \rightarrow f_0^1(t, x, \tilde{p}(t), \dot{\tilde{p}}(t)) \end{aligned}$$

is a normal integrand.

Proposition 5. Let $\tilde{x}(\cdot) \in \mathcal{A}$ be an admissible trajectory of the problem (1) — (5) such that there exists a mapping $\tilde{p}(\cdot) \in AC([t_0, t_1] : R^n)$ satisfying (12) and the following conditions:

$$\sup \{ f_0^1(t, x, \tilde{p}(t), \dot{\tilde{p}}(t)) ; x \in \mathcal{A}(t) \} = f_0^1(t, \tilde{x}(t), \tilde{p}(t), \dot{\tilde{p}}(t)) \text{ a.e.} \quad (18)$$

$$\begin{aligned} \sup \{ g^1(x_0, x_1, \tilde{p}(t_0), \tilde{p}(t_1)) ; (x_0, x_1) \in \mathcal{A}(t_0) \times \mathcal{A}(t_1) \} = \\ = g^1(\tilde{x}(t_0), \tilde{x}(t_1), \tilde{p}(t_0), \tilde{p}(t_1)) \quad (19) \end{aligned}$$

where f_0^1 , g^1 and $\mathcal{A}(\cdot)$ are defined, respectively, by (10), (11) and (3).

Then $\tilde{x}(\cdot)$ is an optimal trajectory of the problem (1) — (5).

Proof. From (9) — (11) it follows that (18) and (19) imply (13) hence the optimality of $\tilde{x}(\cdot)$ follows from Proposition 1.

Remark 6. Some refinements of the conditions (18) and (19) may be given for the problems in which the reachability sets $\mathcal{A}(t)$, $t \in [t_0, t_1]$, may be characterized in terms of properties of the multifunctions $F(\cdot, \cdot)$ and $X(\cdot)$.

Since the reachability sets $\mathcal{A}(t)$ cannot be easily found, except in some types of particular problems, weaker but easier verifiable conditions than (18) and (19) may be preferable in some problems. The simplest way to obtain such conditions is to replace the reachability sets $\mathcal{A}(t)$, $t \in [t_0, t_1]$, on which the supremum is taken in (18) and (19) by the sets $X(t)$ that define the phase space constraints (2).

An immediate corollary of Proposition 5 is the following :

Proposition 7. Let $\tilde{x}(\cdot) \in \mathcal{A}$ be an admissible trajectory of the problem (1) — (5) such that there exists a mapping $\tilde{p}(\cdot) \in AC([t_0, t_1] : R^n)$ satisfying (12) and the following conditions :

$$\sup \{f_0^1(t, x, \tilde{p}(t), \dot{\tilde{p}}(t)); x \in X(t)\} = f_0^1(t, \tilde{x}(t), \tilde{p}(t), \dot{\tilde{p}}(t)) \text{ a.e.} \quad (20)$$

$$\begin{aligned} \sup \{g^1(x_0, x_1, \tilde{p}(t_0), \tilde{p}(t_1); (x_0, x_1) \in X(t_0) \times X(t_1)\} = \\ = g^1(\tilde{x}(t_0), \tilde{x}(t_1), \tilde{p}(t_0), \tilde{p}(t_1)). \end{aligned} \quad (21)$$

Then $\tilde{x}(\cdot)$ is an optimal trajectory of the problem (1) — (5).

Remark 8. In the case (1) — (5) is a convex problem, i.e. the multifunctions $X(\cdot)$, and $F(\cdot, \cdot)$ are convex-valued, the function $g(\cdot, \cdot)$ is convex and for any fixed t and p , $H(t, \cdot, p)$ is a concave function, then conditions (12) and (20) are equivalent with: $(-\dot{\tilde{p}}(t), \dot{\tilde{x}}(t)) \in \partial_{(x, p)} H(t, \tilde{x}(t), \tilde{p}(t))$ and (21) is equivalent with: $(-\tilde{p}(t_0), \tilde{p}(t_1)) \in \partial g(\tilde{x}(t_0), \tilde{x}(t_1))$ where ∂ denotes the subdifferential in the sense of Convex Analysis ([11]).

It follows that in the case of a convex problem the sufficient optimality conditions in Proposition 7 coincide with the sufficient optimality conditions in [11].

We note that, according to the results in [11], under some additional hypotheses, conditions (12), (20) and (21) are also necessary for the optimality of the admissible trajectory $\tilde{x}(\cdot)$ of the convex problem (1) — (5).

We shall prove now that the sufficient optimality conditions for the fixed time-interval problem in [2] imply the conditions in Proposition 5.

In [2] the problem (1) — (5) is considered in the particular case $g(\cdot, \cdot)$ and $f_0(\cdot, \cdot, \cdot)$ are of the forms :

$$g(x_0, x_1) = g_0(x_1), f_0(t, x, v) = f(t, x) \quad (22)$$

and the sufficient optimality conditions are expressed in terms of the support function of the multifunction $F(\cdot, \cdot)$:

$$H_0(t, x, p) = \sup \{\langle p, v \rangle; v \in F(t, x)\} \quad (23)$$

According to the Theorem stated in [2], a sufficient optimality condition for the admissible trajectory $\tilde{x}(\cdot) \in \mathcal{A}$ of the problem (1)–(5), (22), is the existence of a mapping $\psi(\cdot) \in AC([t_0, t_1]: R^n)$ satisfying (12) and the following conditions:

$$\sup \{ \langle \dot{\psi}(t), x \rangle + H_0(t, x, \psi(t)); x \in X(t) \} = \langle \dot{\psi}(t), \tilde{x}(t) \rangle + H_0(t, \tilde{x}(t), \psi(t)) \quad (24)$$

$$\inf \{ \langle \dot{\psi}(t), x \rangle + f(t, x); x \in X(t) \} = \langle \dot{\psi}(t), \tilde{x}(t) \rangle + f(t, \tilde{x}(t)) \quad (25)$$

$$\sup \{ \langle \dot{\psi}(t_0), x_0 \rangle; x_0 \in X(t_0) \} = \langle \dot{\psi}(t_0), \tilde{x}(t_0) \rangle \quad (26)$$

$$\inf \{ \langle \dot{\psi}(t_1), x_1 \rangle + g_0(x_1); x_1 \in X(t_1) \} = \langle \dot{\psi}(t_1), \tilde{x}(t_1) \rangle + g_0(\tilde{x}(t_1)) \quad (27)$$

Proposition 9. If the admissible trajectory $\tilde{x}(\cdot) \in \mathcal{A}$ of the problem (1)–(5), (22), and the mapping $\psi(\cdot) \in AC([t_0, t_1]: R^n)$ satisfy (12) and (24)–(27) then they satisfy also (18)–(19).

Proof. Since in case (22) holds the functions g^1 defined by (11) is given by: $g(x_0, x_1, p_0, p_1) = \langle p_0, x_0 \rangle - \langle p_1, x_1 \rangle - g_0(x_1)$, conditions (26) and (27) coincide with (21) which implies (19).

We shall prove now that (22), (12) and (24) imply:

$$\sup \{ \langle \dot{\psi}(t), x \rangle; x \in \mathcal{A}(t) \} = \langle \dot{\psi}(t), \tilde{x}(t) \rangle \text{ for any } t \in [t_0, t_1] \quad (28)$$

If $x(\cdot) \in A$ then from (12) and (24) it follows:

$$\begin{aligned} \frac{d}{dt} \langle \dot{\psi}(t), x(t) - \tilde{x}(t) \rangle &= \langle \dot{\psi}(t), x(t) \rangle + \langle \dot{\psi}(t), \dot{x}(t) \rangle - \\ &- [\langle \dot{\psi}(t), \tilde{x}(t) \rangle + H_0(t, \tilde{x}(t), \psi(t))] \leq \langle \dot{\psi}(t), x(t) \rangle + H_0(t, x(t), \psi(t)) - \\ &- [\langle \dot{\psi}(t), \tilde{x}(t) \rangle + H_0(t, \tilde{x}(t), \psi(t))] \leq 0 \text{ a.e. on } [t_0, t_1] \text{ hence } t \mapsto \langle \dot{\psi}(t), \\ x(t) - \tilde{x}(t) \rangle &\text{ is a nonincreasing function on } [t_0, t_1] \text{ and since, by (26),} \\ \langle \dot{\psi}(t_0), x(t_0) - \tilde{x}(t_0) \rangle &\leq 0, \text{ then (28) holds.} \end{aligned}$$

From (25) and (28) it follows that $\inf \{ f(t, x); x \in \mathcal{A}(t) \} = f(t, \tilde{x}(t))$ and therefore, from (24) it follows (18) since in the case (22) holds, the functions $H(\cdot, \cdot, \cdot, \cdot)$ and $H_0(\cdot, \cdot, \cdot)$ are related by: $H(t, x, p) = H_0(t, x, p) + f(t, x)$.

We consider now a „parametrized” optimal control problem where instead of trajectories $x(\cdot)$, pairs, $(u(\cdot), x(\cdot))$ of controls and trajectories are considered.

The parametrized control problem we are studying is defined by a parametrized differential equation of the form:

$$\frac{dx}{dt} = f(t, x, u) \quad (29)$$

where $f(\cdot, \cdot, \cdot, \cdot): [t_0, t_1] \times R^n \times R^m \rightarrow R^n$ and by two multifunctions $X(\cdot): [t_0, t_1] \rightarrow \mathcal{P}(R^n)$, $U(\cdot, \cdot): [t_0, t_1] \times R^n \rightarrow \mathcal{P}(R^m)$ that define the phase space and control constraints as follows: for any measurable mapping $u(\cdot): [t_0, t_1] \rightarrow R$ we denote by $\mathcal{A}_{u(\cdot)}$ the (possibly empty) set of solutions $x(\cdot)$ of the differential equation:

$$\frac{dx}{dt} = f(t, x, u(t)) \quad (30)$$

that satisfies (2) and:

$$u(t) \in U(t, x(t)) \text{ a.e. on } [t_0, t_1] \quad (31)$$

We say that a measurable mapping $u(\cdot)$ is an *admissible control* if the set $\mathcal{A}_{u(\cdot)}$ of its corresponding admissible trajectories is not empty. We denote by $\mathcal{U}$ the set of the admissible controls and by $\mathcal{U}_0 = \bigcup \{\mathcal{A}_{u(\cdot)}; u(\cdot) \in \mathcal{U}\}$ the set of all admissible trajectories.

For the existence and uniqueness of the initial value problem for the differential equation (30) for a fixed measurable control $u(\cdot)$ it is enough to assume that for any $x \in R^n$ the mapping $f(\cdot, x, \cdot)$ is a normal integrand and for any $t \in [t_0, t_1]$, $f(t, \cdot, u)$ is locally Lipschitz uniformly with respect to $u \in U(t, x)$, with an integrable Lipschitz constant ([4]). In this case, if $X(t_0)$ is a singleton then for any $u(\cdot) \in \mathcal{U}$ the set $\mathcal{A}_{u(\cdot)}$ of the corresponding trajectories is a singleton.

We may write the problem (29)–(31) as a problem of the form (1)–(2) where the multifunction $F(\cdot, \cdot)$ is given by:

$$F(t, x(t)) = f(t, x, U(t, x)).$$

If $\mathcal{A}$ denotes the set of admissible trajectories of the problem (1), (2), (32) then obviously $\mathcal{A}_0 \subset \mathcal{A}$ and we have $\mathcal{A}_0 = \mathcal{A}$ if, for instance, $f(\cdot, \cdot, \cdot)$ and $U(\cdot, \cdot)$ satisfy the hypotheses of „Filippov's lemma” ([10]) which imply that for any $x(\cdot) \in \mathcal{A}$ and for any measurable mapping $v(\cdot)$ satisfying: $v(t) \in f(t, x(t), U(t, x(t)))$ a.e., there exists a measurable mapping $u(\cdot)$ satisfying (31) and $v(t) = f(t, x(t), u(t))$, a.e.

If $\mathcal{C}$ denotes the set $\{(u(\cdot), x(\cdot)); u(\cdot) \in \mathcal{U}, x(\cdot) \in \mathcal{A}_{u(\cdot)}\}$ of admissible pairs of the problem (29)–(31) then the optimality criterion consists in minimizing a functional $I(\cdot, \cdot): \mathcal{C} \rightarrow R$:

$$I(u(\cdot), x(\cdot)) = g(x(t_0), x(t_1)) + \int_{t_0}^{t_1} f_0(t, x(t), u(t)) dt \quad (33)$$

where, as in problem (1)–(5), the functions $g(\cdot, \cdot): X(t_0) \times X(t_1) \rightarrow R$ and $f_0(\cdot, \cdot, \cdot): [t_0, t_1] \times R^n \times R^m \rightarrow R$ are lower semicontinuous and the last one is a normal integrand.

An admissible pair $(\tilde{u}(\cdot), \tilde{x}(\cdot)) \in \mathcal{C}$ is said to be *optimal* if:

$$I(\tilde{u}(\cdot), \tilde{x}(\cdot)) = \inf \{I(u(\cdot), x(\cdot)); (u(\cdot), x(\cdot)) \in \mathcal{C}\} \quad (34)$$

The following sufficient optimality conditions for the problem defined by (29), (2), (31), (33), (34) may be obtained in the same way as those in Propositions 1, 5, 7 for the problem (1)–(5) were obtained.

The functions H , $\mathcal{H}$ and the multifunction $\tilde{U}(\cdot, \cdot, \cdot)$ (corresponding to $V(\cdot, \cdot, \cdot)$ given by (8)) are defined by:

$$\mathcal{H}(t, x, p, u) = \langle p, f(t, x, u) \rangle - f_0(t, x, u) \quad (35)$$

$$H(t, x, p) = \sup \{ \mathcal{H}(t, x, p, u) ; u \in U(t, x) \} \quad (36)$$

$$\tilde{U}(t, x, p) = \{ u \in U(t, x) ; H(t, x, p) = \mathcal{H}(t, x, p, u) \} \quad (37)$$

and the functional $I_1(\cdot, \cdot) : \mathcal{A}_0 \times AC([t_0, t_1] : R^n) \rightarrow R$ is defined by (9)–(11).

Proposition 10. Let $(\tilde{u}(\cdot), \tilde{x}(\cdot)) \in \mathcal{C}$ be an admissible pair of the problem (29), (2), (31), (33)–(34) such that there exists a mapping $\tilde{p}(\cdot) \in AC([t_0, t_1] : R^n)$ satisfying the following conditions:

$$\tilde{u}(t) \in \tilde{U}(t, \tilde{x}(t), \tilde{p}(t)) \text{ a.e. on } [t_0, t_1] \quad (38)$$

$$I_1(\tilde{x}(\cdot), \tilde{p}(\cdot)) = \sup \{ I_1(x(\cdot), \tilde{p}(\cdot)) ; x(\cdot) \in \mathcal{A}_0 \} \quad (39)$$

Then $(\tilde{u}(\cdot), \tilde{x}(\cdot))$ is optimal.

Proposition 11. Let $(\tilde{u}(\cdot), \tilde{x}(\cdot)) \in \mathcal{C}$ be an admissible pair such that there exists $\tilde{p}(\cdot) \in AC([t_0, t_1] : R^n)$ satisfying (38) and (18), (19) where $\mathcal{A}(t)$ is replaced by $\mathcal{A}_0(t)$ defined as in (3) for any $t \in [t_0, t_1]$.

Then $(\tilde{u}(\cdot), \tilde{x}(\cdot))$ is optimal.

Proposition 12. Let $(\tilde{u}(\cdot), \tilde{x}(\cdot)) \in \mathcal{C}$ be an admissible pair such that there exists $p(\cdot) \in AC([t_0, t_1] : R^n)$ satisfying (38), (20), (21)

Then $(\tilde{u}(\cdot), \tilde{x}(\cdot))$ is optimal.

We remark that Proposition 12 may be obtained from Theorem 1 in [9] but Proposition 11 cannot. The following example from [12] shows that in the nonconvex case the conditions in Proposition 12 could be much weaker than those in Proposition 11.

Let us consider the problem (29), (2), (31), (33)–(34) in the case: $t_0 = 0$, $t_1 = 1$, $X(t) = R$ for $t \in (0, 1]$ and $X(0) = \{0\}$, $U(t, x) = [-1, +1]$, $f(t, x, u) \equiv u$, $g(x_0, x_1) \equiv 0$, $f_0(t, x, u) \equiv -x^2$ for any $t \in [0, 1]$, $x, x_0, \in R$, $u \in [-1, +1]$.

It is easy to see that in this case we have: $\mathcal{H}(t, x, p, u) = p \cdot u + x^2$, $H(t, x, p) = |p| + x^2$, $\tilde{U}(t, x, p) = \{+1\}$ if $p > 0$, $\tilde{U}(t, x, p) = \{-1\}$ if $p < 0$ and $\tilde{U}(t, x, p) = [-1, +1]$ if $p = 0$, $g^1(x_0, x_1, p_0, p_1) = -p_1 \cdot x_1$, $f_0^1(t, x, p, q) = x^2 + qx + |p|$.

Since for any function $p(\cdot) \in AC([0, 1] : R)$ we have:

$\sup \{ f_0^1(t, x, p(t), p(t)) ; x \in R \} = +\infty$, for any $t \in (0, 1)$ hence

Proposition 12 cannot be used to obtain optimal controls of this problem.

Fortunately, in this simple case it is easy to prove that the reachable sets are given by: $\mathcal{A}(t) = \mathcal{A}_0(t) = [-t, +t]$ for any $t \in [0, +1]$ and from (18) and (19) we obtain: if there exists $\tilde{p}(\cdot) \in AC([0, 1]: R)$ such that $\tilde{x}(t) = t$ if $\tilde{p}(t) \leq 0$ and $\tilde{x}(t) = -t$ if $\tilde{p}(t) \geq 0$ for almost any $t \in [0, 1]$ and $\tilde{x}(1) = 1$ if $\tilde{p}(1) \leq 0$ and $\tilde{x}(1) = -1$ if $\tilde{p}(1) \geq 0$ then the admissible pair $(\tilde{u}(\cdot), \tilde{x}(\cdot))$ is optimal. Since $\tilde{u}(\cdot)$ should be absolutely continuous, from the conditions above it follows that we can have either $\tilde{x}(t) = t$ $\tilde{u}(t) = 1$ or $\tilde{x}(t) = -t$ and $\tilde{u}(t) = -1$ for any $t \in [0, 1]$ the function $\tilde{p}(t) \equiv 0$ being the only one that satisfies the conditions in Proposition 11.

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SUR LA REPRÉSENTATION GÉODÉSIQUE ET SUBGÉODÉSIQUE DES ESPACES DE RIEMANN

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Il est bien connu le théorème de Beltrami [1] qui montre que si un espace de Riemann V_n admet une représentation géodésique sur un espace de Riemann $\bar{V}_n$ à courbure constante, l'espace V_n est lui aussi à courbure constante. Siniukov [5] a montré que si un espace de Riemann $V_n (n > 2)$ admet une représentation géodésique non triviale sur un espace $\bar{V}_n$ symétrique, les deux espaces $V_n, \bar{V}_n$ sont à courbure constante. G. Vrănceanu [9] a donné une nouvelle démonstration du théorème de Siniukov. Récemment P. Venzi [7], [8] a obtenu les résultats intéressants sur la représentation géodésique des espaces de Riemann.

Dans le §1, en utilisant les résultats de Venzi, nous obtenons le théorème suivant : *Si une espace de Riemann $V_n (n > 2)$ admet une représentation géodésique non triviale sur une S -variété $\bar{V}_n$ à tenseur de courbure irréductible, alors V_n et $\bar{V}_n$ sont les espaces à courbure constante (théorème 1).*

Dans le §2 nous considérons les espaces de Riemann en représentation subgéodésique et nous obtenons (théorème 2) une extension des théorèmes de Beltrami, de Siniukov et de Venzi.

0. Etant donnés deux espaces de Riemann V_n et $\bar{V}_n$ ayant, respectivement pour métriques :

$$ds^2 = g_{ij} dx^i dx^j, \quad d\bar{s}^2 = \bar{g}_{ij} dx^i dx^j,$$

es deux espaces V_n et $\bar{V}_n$ sont en correspondance géodésique si nous avons es formules de Weyl [9] :

$$(1) \quad \bar{\Gamma}_{jk}^i = |\Gamma_{jk}^i| + \delta_j^i \psi_k + \delta_k^i \psi_j,$$

où $|\Gamma_{jk}^i|$, resp. $|\bar{\Gamma}_{jk}^i|$, sont les symboles de Christoffel de la seconde espace de V_n , resp. $\bar{V}_n$ et où ψ_i sont des dérivées partielles d'une certaine fonction ψ des variables $x^1, \dots, x^n$. On dit que la correspondance géodésique n'est pas triviale si le vecteur ψ_j n'est pas nul, donc si la fonction ψ n'est pas une constante.

En tenant compte de (1) nous avons :

$$(2) \quad \bar{R}_{jkl}^i = R_{jkl}^i + \delta_l^i \psi_{jk} - \delta_k^i \psi_{jl}$$

où ψ_{jk} sont données par les formules :

$$\psi_{jk} = \psi_{j,k} - \psi_j \psi_k; \quad \psi_{j,k} = \frac{\partial \psi_j}{\partial x^k} - \psi_s |\Gamma_{jk}^s|$$

et où R_{jkl}^i et $\bar{R}_{jkl}^i$ sont respectivement les tenseurs de courbure des espaces V_n et $\bar{V}_n$.

Si dans le formule (2) on fait la contraction par rapport à i et k on obtient la formule :

$$(2') \quad \bar{R}_{jl} = R_{jl} - (n - 1) \psi_{jl}$$

ou $R_{jl} = R_{jil}^i$, $\bar{R}_{jl} = \bar{R}_{jil}^i$ sont les tenseurs de Ricci des espaces V_n et $\bar{V}_n$.
L'espace de Riemann $\bar{V}_n$ est S -variété [7] si on a :

$$(3) \quad \bar{R}_{ijk;mr}^h = \bar{R}_{ijk;rm}^h$$

où $\bar{R}_{ijk;mr}^h$ sont les dérivées seconde covariantes du tenseur de courbure $\bar{R}_{ijk}^h$ dans l'espace $\bar{V}_n$.

L'espace $\bar{V}_n$ est à tenseur de courbure irréductible [6] s'il n'existe pas deux vecteurs u^i et v^i noncolinéaires tel que

$$\bar{R}_{ijkh} u^i v^j = 0, \text{ où } \bar{R}_{ijkh} = \bar{g}_{is} \bar{R}_{jkh}^s$$

1. La démonstration du théorème 1. De (3) on a :

$$(4) \quad \bar{R}_{ij;mr} = \bar{R}_{ij;rm}.$$

En tenant compte de (4) il en résulte :

$$(4') \quad \bar{R}_{js} \bar{R}_{irm}^s + \bar{R}_{is} \bar{R}_{jrm}^s = 0$$

En utilisant les relations (2), (2') et (4') on obtient :

$$R_{js} R_{irm}^s + R_{is} R_{jrm}^s + R_{jm} \psi_{ir} + R_{im} \psi_{jr} - R_{jr} \psi_{im} - R_{ir} \psi_{jm} - \\ - (n - 1) (\psi_{js} R_{irm}^s - \psi_{is} R_{jrm}^s) = 0$$

En multipliant ces formules par g^{ij} et en faisant la somme nous obtenons :

$$(4'') \quad \psi_{sj} R_i^s - \psi_{si} R_j^s = 0, \text{ où } R_j^i = g^{ir} R_{rj}$$

En utilisant les relations (2) et (3) on obtient :

$$R_{ihj}^s R_{smr}^h + R_{imr}^s R_{sjk}^h + R_{jmr}^s R_{ish}^h + R_{kmr}^s R_{ijs}^h + \delta_r^h R_{ikj}^s \psi_{sm} - \\ - \delta_m^h \psi_{sr} R_{ikj}^s + \psi_{im} R_{rjk}^h - \psi_{ir} R_{mjk}^h + \delta_k^h \psi_{js} R_{imr}^s - \delta_j^h \psi_{sk} R_{imr}^s + \\ (5) \quad + \psi_{jm} R_{irk}^h - \psi_{jr} R_{imk}^h + \delta_k^h \psi_{is} R_{jmr}^s + \psi_{km} R_{ijr}^h - \psi_{kr} R_{ijm}^h - \delta_j^h \psi_{is} R_{kmr}^s = 0$$

En multipliant les relations (5) par g^{kr} et en sommant, nous obtenons :

$$(6) \quad \begin{aligned} & g^{kr}(R_{ikj}^s R_{hsmr} + R_{imr}^s R_{hsjk} + R_{jmr}^s R_{hisk} + R_{kmr}^s R_{hij s}) + R_{ihj}^s \psi_{sm} - \\ & - g_{hm} g^{kr} R_{ikj}^s \psi_{sr} + \psi_{im} R_{jh} - \psi_{is} R_{jmh}^s + \psi_{js} R_{imh}^s - g_{jh} g^{kr} \psi_{sk} R_{imr}^s - \\ & - \psi_{js} R_{mih}^s + \psi_{is} R_{jmh}^s + \psi_{ms} R_{jih}^s - f R_{hijm} - g_{jh} \psi_{is} g^{sr} R_{rm} = 0, \end{aligned}$$

où $f = g^{ij} \psi_{ij}$ et $R_{hsmr} = g_{hi} R_{smr}^i$.

Si l'on somme la formule (6) avec la même formule où l'on change h par i on obtient :

$$(7) \quad \begin{aligned} & \psi_{sm} R_{ihj}^s + \psi_{sm} R_{hij}^s - g_{hm} g^{kr} \psi_{sr} R_{ikj}^s - \psi_{sr} g_{im} g^{kr} R_{hjk}^s + R_{jh} \psi_{im} + \\ & + R_{ij} \psi_{mh} + \psi_{js} R_{imh}^s + \psi_{is} R_{hmi}^s - g_{jh} g^{kr} \psi_{ks} R_{imr}^s - \\ & - g_{ij} g^{kr} \psi_{ks} R_{hmr}^s - g_{jh} \psi_{is} g^{sr} R_{rm} - g_{ij} \psi_{hs} g^{sr} R_{rm} = 0 \end{aligned}$$

Si l'on somme la formule (7) avec la même formule où l'on change j par m on obtient :

$$(8) \quad \begin{aligned} & R_{jh} \psi_{im} + R_{ij} \psi_{hm} - g_{jh} \psi_{is} R_m^s + R_{hm} \psi_{ij} - g_{ij} \psi_{sh} R_m^s + \\ & + R_{im} \psi_{hj} - g_{mh} \psi_{is} R_j^s - g_{im} \psi_{sh} R_j^s = 0 \end{aligned}$$

En multipliant les formules (8) par g^{ij} et en sommant, nous obtenons :

$$(9) \quad (n+1) \psi_{hs} R_m^s - R \psi_{hm} - f R_{hm} + g_{hm} \psi_{sr} R^{sr} - \psi_{sm} R_h^s = 0$$

où l'on a noté $R^{ij} = g^{ir} R_r^j$, $R = g^{ij} R_{ij}$.

En multipliant les relations (9) par g^{mh} , en sommant et en employant (4'') il en résulte $Rf = n R^{ij} \psi_{ij}$. Alors, de (9) nous avons :

$$(9') \quad \psi_{sh} R_m^s = \frac{f}{n} R_{mh} - \frac{Rf}{n^2} g_{mh} + \frac{R}{n} \psi_{mh} = \psi_{ms} R_h^s$$

En tenant compte des relations (9'), les formules (8) deviennent :

$$(8') \quad \begin{aligned} & (fg_{hm} - n \psi_{hm}) (n R_{ij} - R g_{ij}) + (fg_{ij} - n \psi_{ij}) (n R_{hm} - R g_{hm}) + \\ & + (fg_{jm} - n \psi_{jm}) (n R_{ih} - R g_{ih}) + (fg_{ih} - n \psi_{ih}) (n R_{jm} - R g_{jm}) = 0 \end{aligned}$$

pour tout les indices $h, m, i, j \in \{1, 2, \dots, n\}$.

En particulier, pour $h = j$ et $m = i$ les formules (8') nous donnent

$$(8'') \quad (fg_{ij} - n\psi_{ij})(nR_{ij} - Rg_{ij}) = 0$$

ou i, j sont des indices fixes.

Le premier membre de (8') est symétrique dans h et m et aussi dans i et j . De (8') on a :

$$\begin{aligned} & (fg_{jm} - n\psi_{jm})(nR_{ih} - Rg_{ih}) + (fg_{ih} - n\psi_{ih})(nR_{jm} - Rg_{jm}) - \\ & = (fg_{jh} - n\psi_{jh})(nR_{im} - Rg_{im}) + (fg_{im} - n\psi_{im})(nR_{ih} - Rg_{jh}) = \\ (8''') & = (fg_{hm} - n\psi_{hm})(nR_{ij} - Rg_{ij}) + (fg_{ij} - n\psi_{ij})(nR_{hm} - Rg_{hm}) \end{aligned}$$

Les relations (8'), (8'') et (8''') nous donnent :

$$(10) \quad (n\psi_{ij} - fg_{ij})(nR_{mh} - Rg_{mh}) = 0$$

pour tout $i, j, h, m \in \{1, 2, \dots, n\}$. Puisque la correspondance géodésique des espaces $V_n, \bar{V}_n$ est non triviale et l'espace $\bar{V}_n$ est à tenseur de courbure irréductible nous avons :

$$(10') \quad n\psi_{ij} - fg_{ij} \neq 0$$

En effet, les relations $n\psi_{ij} - fg_{ij} = 0$ conduisent à

$$(11) \quad \bar{R}_{jkl}^i = R_{jkl}^i + \frac{f}{n} (\delta_{il}^i g_{jk} - \delta_k^i g_{jl})$$

En tenant compte de (11) on a :

$$g_{ih} \bar{R}_{jkl}^h - g_{jh} \bar{R}_{ikl}^h = 0$$

Puisque l'espace $\bar{V}_n$ est à tenseur de courbure irréductible il résulte que le système

$$(12) \quad y_{ih} \bar{R}_{jkl}^h + y_{jh} \bar{R}_{ikl}^h = 0$$

admet une solution unique, à un facteur près. On peut remarquer que le système (12) est vérifié par g_{ij} et aussi par $\bar{g}_{ij}$. Nous avons donc

$$(13) \quad \bar{g}_{ij} = e^{2u} g_{ij}$$

où u est une fonction arbitraire. On sait que ([4], Theoreme 1) deux métriques conformes possèdent les mêmes géodésiques si et seulement si les deux métriques sont homothétiques. Nous avons donc $u = \text{constant}$

Alors les relations (13) implique $|\bar{jk}| = |jk|$ et en tenant compte de (1) on obtient

$$\delta_j^i \psi_k + \delta_k^i \psi_j = 0$$

En contractant par rapport à i et j nous obtenons $(n+1) \psi_k = 0$, c'est-à-dire $\psi_k = 0$.

Il résulte que la correspondance géodésique des espaces V_n et $\bar{V}_n$ est triviale, ce qui contredit l'hypothèse du théorème. Donc, on a (10').

De (10) et (10') nous avons

$$(14) \quad R_{ij} = \frac{R}{n} g_{ij},$$

c'est-à-dire V_n est une espace d'Einstein. Les formules (14) implique (P. Venzi [7], p. 196)

$$(14') \quad (n\psi_{im} - fg_{im}) P_{hrjk} = 0$$

ou

$$P_{jkh}^i = R_{jkh}^i - \frac{\delta_k^i}{n-1} R_{jh} + \frac{\delta_h^i}{n-1} R_{jk}, \quad P_{ijkh} = g_{is} P_{sjkh}$$

De (10') et (14') on obtient :

$$(11'') \quad P_{hrjk} = 0$$

De (14) et (11'') il résulte que V_n est à courbure constante. En utilisant le théorème de Beltrami il en résulte que l'espace $\bar{V}_n$ est aussi à courbure constante.

2. *Une théorème sur la représentation subgéodésique des espaces de Riemann.* Considérons deux espaces de Riemann V_n , $\bar{V}_n$ ayant, respectivement, pour métrique $ds^2 = g_{ij} dx^i dx^j$, $d\bar{s}^2 = \bar{g}_{ij} dx^i dx^j$ et soit ξ^i un champ de vecteurs. Soient $|jk|$, resp. $|\bar{jk}|$, les symboles de Christoffel de la seconde espace de V_n , resp. $\bar{V}_n$. Les espaces V_n et $\bar{V}_n$ ont les mêmes ξ^i -subgéodésiques si et seulement si nous avons les formules de Yano [10], [2]:

$$(15) \quad |\bar{jk}| = |jk| + \delta_j^i \omega_k + \delta_k^i \omega_j + \varphi_{jk} \xi^i$$

où ω_i sont les composantes d'un vecteur covariant et $\varphi_{ij} = \varphi_{ji}$ sont les composantes d'un tenseur covariant symétrique. Nous dirons encore que les espaces V_n et $\bar{V}_n$ sont en représentation ou en correspondance sub-subgéodésique.

Dans l'article [2] nous donnons certaines propriétés des espaces de Riemann V_n et $\bar{V}_n$ subgéodésiquement applicables dans le cas $\varphi_{ij} = g_{ij}$, c'est-à-dire ;

$$(16) \quad |\bar{jk}| = |jk| + \delta_j^i \omega_k + \delta_k^i \omega_j + g_{jk} \xi^i$$

Nous dirons que les espaces V_n et $\bar{V}_n$ qui satisfont (16) sont en représentation subgéodésique non triviale si le vecteur $\omega_i + \xi_i$ n'est pas nu (on a noté $\xi_i = g_{ij}\xi^j$). Nous avons le théorème suivant :

Théorème 2. Soient $V_n(g_{ij})$ et $\bar{V}_n(\bar{g}_{ij})$ ($n > 2$) deux espaces de Riemann en représentation subgéodésique non triviale. Alors :

i) *Il existe une fonction ξ des variables $x_1, \dots, x^n$ tel que*

$$\xi_i = \frac{\partial \xi}{\partial x^i}$$

ii) *Si l'espace $\bar{V}_n(\bar{g}_{ij})$ est à courbure constante, alors l'espace $\tilde{V}_n(e^{-2\xi}g_{ij})$ est lui aussi à courbure constante.*

iii) *Si $\bar{V}_n(\bar{g}_{ij})$ est un espace symétrique dans le sens de Cartan, alors les espaces $\bar{V}_n(\bar{g}_{ij})$ et $\tilde{V}_n(e^{-2\xi}g_{ij})$ sont à courbure constante.*

iv) *Si $\bar{V}_n(\bar{g}_{ij})$ est un espace recurrent alors les espaces $\bar{V}_n(\bar{g}_{ij})$ et $\tilde{V}_n(e^{-2\xi}g_{ij})$ sont à courbure constante.*

v) *Si $\bar{V}_n(\bar{g}_{ij})$ est S-variété à tenseur de courbure irréductible, alors les espaces $\bar{V}_n(\bar{g}_{ij})$ et $\tilde{V}_n(e^{-2\xi}g_{ij})$ sont à courbure constante.*

Démonstration.

i) Voir [2].

ii) Si nous notons $\tilde{g}_{ij} = e^{-2\xi}g_{ij}$ alors nous avons :

$$(17) \quad |\tilde{g}_{jk}| = |g_{jk}| - \delta_j^i \xi_k - \delta_k^i \xi_j + g_{jk} \xi^i$$

De (16) et (17) on a

$$|\tilde{g}_{jk}| = |g_{jk}| + \delta_j^i \psi_k + \delta_k^i \psi_j$$

où $\psi_k = \omega_k + \xi_k$. Ces formules nous montrent que les espaces de Riemann $\tilde{V}_n(\tilde{g}_{ij})$ et $\bar{V}_n(\bar{g}_{ij})$ sont en représentation géodésiques. Dans la suite on utilise le théorème de Beltrami [1], [9].

iii) On utilise le théorème de Siniukov [5] (voir et [9]).

iv) On utilise le théorème de Venzi ([7] Theoreme [5]) (voir et [3]).

v) On utilise le résultat de §1 (le théorème 1).

Remarque. Si dans (16) nous avons $\xi^i = 0$, alors les affirmations ii), iii), iv) se réduisent respectivement aux théorèmes de Beltrami [1], de Siniukov [5] et de Venzi [7].

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UNE CLASSE D'ANNEAUX PRESQUE LASKÉRIENS

NICOLAE RADU

Dans [3] nous avons démontré le théorème suivant :

Théorème. Soient k un corps commutatif de caractéristique $p > 0$ et A une k -algèbre commutative locale noethérienne d'idéal maximal M . Alors les affirmations suivantes sont équivalentes :

- a) A est une k -algèbre formellement lisse pour la topologie M -adique.
- b) L'anneau $A \otimes_k k^{p^{-1}}$ est un anneau noethérien régulier.

Soit $\bar{k} = \bigcup_{n=0}^{\infty} k^{p^{-n}}$. Dans les conditions a) et b) du théorème précédent l'anneau $\bar{A} = A \otimes_k \bar{k}$ est-il un anneau noethérien?

Nous montrons, dans la suite, que si, de plus, le corps résiduel de A est une extension finie d'une extension séparable de k , alors A est un anneau noethérien régulier. Ce résultat s'obtient comme un corollaire d'un théorème qui donne une condition dans laquelle une limite inductive d'anneaux noethériens est un anneau presque laskérien. La proposition 1 donne une caractérisation des anneaux presque laskériens avec la condition maximale des chaînes pour les idéaux radicaux.

Rappelons qu'un anneau commutatif A est appelé presque laskérien si tout idéal propre de type fini I de A est une intersection finie des idéaux premiers. Dans la suite, les anneaux sont supposés commutatifs et unitaires. On dit qu'un idéal I d'un anneau A est radical s'il est égal à sa racine $r(I)$.

Proposition 1. Soit A un anneau tel que toute suite ascendente d'idéaux radicaux de A est stationnaire. Alors les affirmations suivantes sont équivalentes :

- a) A est un anneau presque laskérien.
- b) Pour tout idéal I de A de type fini tel que $r(I) = P$ soit un idéal premier, il existe un élément $a \in A \setminus P$ tel que $I : Aa$ soit un idéal P -primaire.
- c) Pour tout idéal propre I de type fini de A et P un idéal premier $P \supseteq I$ et minimal avec cette propriété, il existe $a \in A \setminus P$ tel que $I : Aa$ soit un idéal P -primaire.

Démonstration. a) $\Rightarrow$ b) Soit I un idéal de type fini de A tel que $r(I) = P$ soit premier. De a) on déduit qu'il existe une décomposition primaire finie et réduite $I = \bigcap_{i=1}^n Q_i$, où Q_i sont des idéaux P_i -primaires, $P_i \neq P_j$, pour $i \neq j$, $i, j = 1, 2, \dots, n$. Puisque $r(I) = P$, il en résulte qu'il existe $i \in \mathbb{N}$, $1 \leq i \leq n$, tel que $P_i = P$. On peut supposer que $i = 1$ et on a $P_i \supset P_1$, pour $i = 2, \dots, n$. Alors $P_1 \not\supseteq \bigcap_{i=2}^n Q_i$, donc il existe un élément $a \in \bigcap_{i=2}^n Q_i$ tel que $a \notin P_1$. Alors $I : Aa = Q_1$ est un idéal P -primaire.

b) $\Rightarrow$ c) Soient I un idéal propre de type fini de A et P un idéal premier $P \supseteq I$ et minimal avec cette propriété. On a $r(I) = P \cap P_1 \cap \dots \cap P_n$, où P_i sont des idéaux premiers de A qui contiennent I et minimaux avec cette propriété, $P_i \neq P$, $P_i \neq P_j$, pour $i \neq j$. Alors il existe $z \in \bigcap_{i=1}^n P_i$ et $z \notin P$. Puisque dans A toute suite ascendente d'idéaux radicaux est stationnaire, on conclut qu'il existe des éléments $a_1, \dots, a_t \in A$ tels que $r\left(\sum_{i=1}^t Aa_i\right) = P$. Puisque $za_i \in r(I)$, il existe $m_i \in \mathbb{N}$ tels que $(za_i^{m_i}) \in I$. Soit $m = \max_{1 \leq i \leq t} m_i$. Alors $a_i^m \in I : Az$. Donc $r(I : Az) = P$ et de b) on obtient qu'il existe un élément $x \in A \setminus P$ tel que $(I : Az) : Ax$ soit P -primaire. Puisque $xz \notin P$, la condition c) est vérifiée.

c) $\Rightarrow$ a) Supposons que A n'est pas presque laskérien. Alors dans l'ensemble des idéaux radicaux il existe un élément maximal pour la propriété qu'il existe un idéal de type fini I tel que $r(I) = Q$ et I n'a pas une décomposition primaire finie. Soient P un idéal premier de A minimal dans l'ensemble des idéaux premiers de A qui contiennent I et $a \in A \setminus P$ tel que $I : Aa$ soit P -primaire. On a alors $I = (I + Aa) \cap (I : Aa)$ et $P \not\supseteq (I + Aa)$, donc $r(I + Aa) \supset r(I)$. Alors l'idéal $I + Aa$ a une décomposition primaire finie, donc I a une décomposition primaire finie, contradiction.

Lemme 1. Soit $A \subseteq B$ une extension d'anneaux. Supposons que pour tout $x \in B$ il existe un entier $n \in \mathbb{N}$ tel que $x^n \in A$. Alors si dans l'anneau A toute suite d'idéaux radicaux de A est stationnaire, l'anneau B a la même propriété.

Démonstration. Il suffit de prouver que pour tout idéal premier Q de B il existe un idéal de type fini I de B tel que $r(I) = Q$. Soit $P = A \cap Q$. Par hypothèse, il existe un idéal de type fini I' de A tel que $r(I') = P$. Alors on a $r(I'B) = Q$.

Lemme 2. Soient $u : A \rightarrow B$ un morphisme fidèlement plat d'anneaux, Q un idéal premier dans B , $P = Q \cap A$ et I un idéal P -primaire dans A . Si pour tout élément $x \in B$ il existe $n \in \mathbb{N}$ tel que $x^n \in A$, alors IB est un idéal Q -primaire dans B .

Démonstration. Évidemment $r(IB) = Q$. Soient $\alpha, \beta \in B$ tel que $\alpha\beta \in IB$ et $\alpha \notin Q$. Il existe $n \in \mathbb{N}$ tel que $\alpha^n \in A \setminus P$. Puisque u est plat, on déduit que α n'est pas un diviseur de zéro dans B/IB , donc α n'est pas un diviseur de zéro dans B/IB et $\beta \in IB$.

Théorème 1. Soient $\{A_i, i \in \Lambda, u_{ij}, i \leq j\}$ un système inductif filtrant d'anneaux noethériens tel que u_{ij} soient fidèlement plats pour $i \leq j$ et pour tout élément $x \in A_j$ il existe $n \in \mathbb{N}$ tel que $x^n \in A_i$. Alors $\bar{A} = \varinjlim A_i$ est un anneau presque laskérien.

Démonstration. Du lemme 1 on obtient que dans $\bar{A}$ toute suite ascendente d'idéaux radicaux est stationnaire. Alors on peut appliquer la proposition 1, donc il suffit de prouver que pour un idéal propre de type fini

$\bar{I}$ de $\bar{A}$ tel que $r(\bar{I}) = \bar{P}$ est un idéal premier, il existe un élément $a \in \bar{A} \setminus \bar{P}$ tel que $\bar{I} : \bar{A}a$ soit un idéal $\bar{P}$ -primaire. Puisque $\bar{I}$ est un idéal de type fini il existe $i \in \Lambda$ tel que pour $I_i = \bar{I} \cap A_i$ on a $I_i \bar{A} = \bar{I}$. Soit $P_i = \bar{P} \cap A_i$, on a $r(I_i) = P_i$ et du fait que A_i est un anneau noethérien s'obtient qu'il existe $a \in A_i \setminus P_i$ tel que $I_i : A_i a$ soit P_i -primaire. Alors du lemme 2 on déduit que $(I_i : A_i) \bar{A} = I_i \bar{A} : \bar{A}a$ est un idéal $\bar{P}$ -primaire dans $\bar{A}$.

Corollaire 1. Soient A un anneau noethérien régulier de corps de fractions K de caractéristique $p > 0$, $K' = \bigcup_{n \geq 0} K^{p^{-n}}$, A' la clôture intégrale de A dans K' . Alors A' est un anneau presque laskérien.

En effet, si A est un corps, l'affirmation est évidente. Dans le cas contraire on a la suite strictement ascendante des corps $K \subset K^{p^{-1}} \subset \dots \subset K^{p^{-n}} \subset \dots$ puisque A est un anneau noethérien. Si A_i est la clôture intégrale de A dans $K^{p^{-i}}$ on a $\bigcup_{i \geq 0} A_i = A'$ et l'inclusion canonique

$A_i \rightarrow A_{i+1}$ est un morphisme fidèlement plat pour tout $i \in \mathbb{N}$, puisque A_i est un anneau régulier. Alors on peut appliquer le théorème précédent et s'obtient que A' est un anneau presque laskérien.

Remarquons que l'anneau A' du corollaire précédent n'est pas noethérien, puisque le corps des fractions K' de A' est un corps parfait.

Corollaire 2. Soient k un corps de caractéristique $p > 0$, A une k -algèbre noethérienne et k' un corps extension de k contenu dans $k^{p^{-\infty}}$. Alors $A \otimes_k k'$ est un anneau presque laskérien.

En effet, on a $k' = \varinjlim k'_i$, où k'_i sont des corps, extensions finies de k contenus dans k' . Alors $A \otimes_k k' = \varinjlim A \otimes_k k'_i$ et pour $k'_i \subseteq k'_j$ le morphisme canonique $A \otimes_k k_i \rightarrow A \otimes_k k_j$ est fidèlement plat, donc l'affirmation s'obtient du théorème.

Corollaire 3. Soient k un corps de caractéristique $p > 0$, A une k -algèbre locale noethérienne de corps résiduel K extension finie d'une extension séparable de k et k' un corps, extension de k contenu dans $k^{p^{-\infty}}$. Alors l'anneau $\bar{A} = A \otimes_k k'$ est un anneau local noethérien.

En effet, il suffit de voir que l'anneau $K \otimes_k k' \simeq K \otimes_A \bar{A}$ est noethérien, donc l'idéal maximal de $\bar{A}$ est de type fini et d'appliquer la proposition suivante.

Proposition 2. (Voir [1] [4]). Un anneau local presque laskérien de l'idéal maximal de type fini est noethérien.

Remarque. Dans le corollaire 3 la condition sur K d'être une extension finie d'une extension séparable de k est nécessaire pour que l'anneau $A \otimes_k k'$ soit un anneau noethérien.

En effet, soit k un corps de caractéristique $p > 0$ qui n'est pas parfait. Alors $\dim_k k^{p^{-\infty}} = \infty$ et $k^{p^{-\infty}} \otimes_R k^{p^{-\infty}}$ n'est pas un anneau noethérien.

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ON CONVERGENT SEQUENCES OF MARKOV PROCESSES

GHEORGHE RĂUȚU

Let E be any Hausdorff space having at least one convergent sequence with distinct terms. Then there exist a sequence $((X(n), Y(n), Z(n)))_{n \geq 1}$ of E^3 -valued random variables and another E^3 -valued random variable (X, Y, Z) on some probability space, $(\Omega, \mathcal{F}, P)$, such that: (i) for each $n > 1$, $X(n)$ and $Z(n)$ are conditionally independent given $Y(n)$; (ii) $(X(n), Y(n), Z(n)) \rightarrow (X, Y, Z)$ P -a.s. as $n \nearrow \infty$; (iii) X and Z are not conditionally independent given Y .

The summary assertion will be proved by constructing a concrete example. Firstly, we give such an example for a particular denumerable state space, and then we „embed” this example in the general setting.

Let $\bar{E} = \{1 + 1/n \mid n = 1, 2, \dots\} \cup \{0, 1\}$ be endowed with the relative topology induced by the usual one for the real line. Let also x, y, z, w be four $\mathbb{R}$ -valued random variables on some probability space, $(\Omega, \mathcal{F}, P)$, such that $x, (y, w)$ and z are independent and, in addition,

$$P \circ x^{-1} = P \circ y^{-1} = P \circ z^{-1} = 1/2 (\varepsilon_0 + \varepsilon_1); \quad P \circ w^{-1} = \sum_{k=1}^{\infty} 1/2^k \varepsilon_{1/k},$$

where $P \circ f^{-1}$ denotes the distribution of f under P and ε_a stands for the Dirac measure carried by $\{a\}$.

For each $n \geq 1$, we define $(\bar{X}(n), \bar{Y}(n), \bar{Z}(n)) : \Omega \rightarrow \bar{E}^3$ as follows

$$\begin{aligned} (1) \quad \bar{X}(n) &= x; & (2) \quad \bar{Y}(n) &= \begin{cases} y & \text{on } (x = 0), \\ 1 + \min(1/n, w) & \text{on } (x = 1); \end{cases} \\ (3) \quad \bar{Z}(n) &= \begin{cases} z & \text{on } (\bar{Y}(n) = 0), \\ \min(1 + 1/n, \bar{Y}(n) + z) & \text{on } (1 \leq \bar{Y}(n) < 1 + 1/n), \\ 1 + z & \text{on } (Y(n) = 1 + 1/n). \end{cases} \end{aligned}$$

(All random variables are defined only almost everywhere.)

Next, consider the random variable $(\bar{X}, \bar{Y}, \bar{Z}) : \Omega \rightarrow \bar{E}^3$, where

$$\begin{aligned} (4) \quad \bar{X} &= x; & (5) \quad \bar{Y} &= \begin{cases} y & \text{on } (x = 0), \\ 1 & \text{on } (x = 1); \end{cases} \\ (6) \quad \bar{Z} &= \begin{cases} z & \text{on } (x = 0, y = 0), \\ 1 & \text{on } (x = 0, y = 1), \\ 1 + z & \text{on } (x = 1). \end{cases} \end{aligned}$$

Let us now point out that $((\bar{X}(n), \bar{Y}(n), \bar{Z}(n)))_{n \geq 1}$ and $(\bar{X}, \bar{Y}, \bar{Z})$ satisfy (i) – (iii), with $\bar{E}$ instead of E .

By making use of (3), we see that the independence between (x, y, w) and z , together with the (x, y, w) – measurability of $(\bar{X}(n), \bar{Y}(n))$ imply the equations

$$P(\bar{Z}(n) \mid (\bar{X}(n), \bar{Y}(n))) = P(\bar{Z}(n) \mid \bar{Y}(n)) ; n \geq 1,$$

where $P(f \mid g)$ denotes the conditional distribution of f given g . That is, $\bar{X}(n)$ and $\bar{Z}(n)$ are conditionally independent given $\bar{Y}(n)$.

The convergence $(\bar{X}(n), \bar{Y}(n)) \rightarrow (\bar{X}, \bar{Y})$ P -a.s. is obvious. As for $\bar{Z}(n)$, whatever $n \geq 1$, we have

$$(7) \quad (\bar{Y}(n) = 0) = (x = 0, y = 0),$$

$$(8) \quad (1 \leq \bar{Y}(n) < 1 + 1/n) = (x = 0, y = 1) \cup (x = 1, w < 1/n),$$

$$(9) \quad (\bar{Y}(n) = 1 + 1/n) = (x = 1, w \geq 1/n).$$

From (7), (8) and (9), it follows that

$$(10) \quad \bar{Z}(n) = \begin{cases} z & \text{on } (x = 0, y = 0), \\ \min(1 + 1/n, 1 + z) & \text{on } (x = 0, y = 1), \\ \min(1 + 1/n, 1 + w + z) & \text{on } (x = 1, w < 1/n), \\ 1 + z & \text{on } (x = 1, w \geq 1/n); \end{cases}$$

for every $n \geq 1$. Taking into account that $(w < 1/n) \searrow \emptyset$ P -a.s. as $n \nearrow \infty$, we infer from (10) that $\bar{Z}(n) \rightarrow \bar{Z}$ P -a.s.

To prove that $\bar{X}$ and $\bar{Z}$ are not conditionally independent given $\bar{Y}$, it suffices to check, for instance, that

$$P(\bar{Z} = 1 \mid \bar{X} = 0, \bar{Y} = 1) \neq P(\bar{Z} = 1 \mid \bar{Y} = 1).$$

Indeed, it is not difficult to prove, by using the independence between x and (y, w) , that $P(\bar{Z} = 1 \mid \bar{Y} = 1) = 2/3$ while $P(\bar{Z} = 1 \mid \bar{X} = 0, \bar{Y} = 1) = 1$.

Let now E be any topological space as in the summary and $(a(n))_{n \geq 1}$ the mentioned convergent sequence, with distinct terms in E . Denoting with a the limit of this sequence, we may suppose that $a \neq a(n)$ for all $n \geq 1$, and, in addition, that there exists $e \in E$ which does not belong to $\{a(n) \mid n \geq 1\} \cup \{a\}$. Let us now consider the continuous injective mapping $f: \bar{E} \rightarrow E$ defined by

$$f(0) = e, f(1) = a, f(1 + 1/n) = a(n); n = 1, 2, \dots$$

and put

$$(X, Y, Z) = (f(\bar{X}), f(\bar{Y}), f(\bar{Z})),$$

$$(X(n), Y(n), Z(n)) = (f(\bar{X}(n)), f(\bar{Y}(n)), f(\bar{Z}(n))); n = 1, 2, \dots$$

It is now immediate that all required conditions (i) — (iii) are fulfilled.

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and the

$$f(x) = \frac{1}{2} (f(x) + f(x))$$

$$f(x) = \frac{1}{2} (f(x) + f(x))$$

It is now necessary to show that the conditions (1) and (2) are satisfied. Let us assume that the conditions (1) and (2) are satisfied. Then we have

Let us assume that the conditions (1) and (2) are satisfied. Then we have

ON APPROXIMATION IN QUADRATIC MEAN FOR THE SOLUTIONS OF TWO PARAMETER STOCHASTIC DIFFERENTIAL EQUATIONS IN HILBERT SPACES

CONSTANTIN TUDOR and MARIA TUDOR

Consider a two parameter stochastic differential equation of the form

$$(1) \quad dx_{s,t} = a(s,t,x) dM_{s,t} + b(s,t,x) dV_{s,t}$$

where $(M_{s,t})$ is a continuous square integrable strong martingale with values in a Hilbert space H , $(V_{s,t})$ is a H -valued continuous process with bounded variation and a, b are lipschitzian functions defined on $[0,1]^2 \times D([0,1]^2, H)$ with values in the space of Hilbert-Schmidt operators on H .

We give bounds for $E(\sup_{u \leq z} |x_u - x_u^n|^2)$, where x^n is an approximant process obtained from the solution x of the equation (1) by a determinist discretization of time.

1. Two parameter strong martingales with values in a Hilbert space and stochastic integrals.

Let H be a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$ and let $(\Omega, \mathcal{F}, P, \mathcal{F}_{s,t}; 0 \leq s, t \leq 1)$ be a stochastic basis. We assume that $\mathcal{F}_{0,0}$ contains the negligible sets from $\mathcal{F}$. We denote:

$$I = [0, 1]^2; I_0 = [0, 1] \times \{0\} \cup \{0\} \times [0, 1], J = (0, 1]^2$$

$$(z, z') = (s, s'] \times (t, t'] \text{ if } z = (s, t), z' = (s', t').$$

$$\mu((z, z')) = \text{the Lebesgue measure of } (z, z').$$

$f((z, z')) = \square_{s',t'} f_{s,t} = f_{s',t'} - f_{s',t} - f_{s,t'} + f_{s,t}$ if $(f_{s,t})$ is a H -valued process and $s \leq s', t \leq t'$.

$$\overline{\mathcal{F}}_{t,t} = \mathcal{B}(\mathcal{F}_{s,1} \cup \mathcal{F}_{1,t}).$$

$\mathcal{L}(H)$ = the Banach space of all continuous linear operators on H .

$\mathcal{L}_2(H)$ = the Hilbert space of all Hilbert-Schmidt operators on H .

Norms in $\mathcal{L}(H)$, $\mathcal{L}_2(H)$ are denoted by $\|\cdot\|, \|\cdot\|_2$.

$L_2(\Omega, \mathcal{F}, P, H)$ = the Hilbert space of all H -valued functions on $(\Omega, \mathcal{F}, P)$ which are square integrable in Bochner sense.

A H -valued process $(M_z)_{z \in I}$ is called square integrable strong martingale (or difference martingale) if:

(i₁) for every $z, M_z \in L_2(\Omega, \mathcal{F}_z, P, H)$.

(i₂) $(M_{s,0}, \mathcal{F}_{s,1}), (M_{0,t}, \mathcal{F}_{1,t})$ are H -valued martingales with one parameter.

(i₃) $E[\square_{s',t'} M_{s,t} | \overline{\mathcal{F}}_{s,t}] = 0$ if $s \leq s', t \leq t'$.

Let (M_z) be a continuous square integrable strong martingale and (e_n) be an orthonormal basis in H .

For every n the process $M_z^n = \langle e_n, M_z \rangle$ is a real two parameter strong martingale, so that by a result of Gihman (see [1]) we find an unique process $(\langle M^n \rangle_z)$, called the characteristic or the quadratic variation of (M_z^n) , with properties :

(j₁) $\langle M^n \rangle$ has continuous paths and is $\mathcal{F}_z$ -adapted.

(j₂) $\langle M^n \rangle$ vanishes on I_0 .

(j₃) $E[(\square_{s',t'} M_{s,t}^n)^2 | \bar{\mathcal{F}}_{s,t}] = E[\square_{s',t'} \langle M^n \rangle_{s,t} | \bar{\mathcal{F}}_{s,t}]$ for every $s < s', t < t'$.

(j₄) $\langle M^n \rangle_{s,t} = P. \lim_{|\lambda| \rightarrow 0} \sum_{i,j=0}^{p,q} (\square_{s_{i+1}, t_{j+1}} M_{s_i, t_j}^n)^2$, where λ is a partition of $[0, s] \times [0, t]$ of the form $0 = s_0 < \dots < s_{p+1} = s, 0 = t_0 < \dots < t_{q+1} = t$ and $|\lambda|$ is the norm of λ .

It is easy to check that the process $\langle M \rangle_z = \sum_{n=1}^{\infty} \langle M^n \rangle_z$ is well defined, satisfies (j₁), (j₂) and moreover

$$E[\square_{s',t'} M_{s,t}^2 | \bar{\mathcal{F}}_{s,t}] = E[\square_{s',t'} \langle M \rangle_{s,t} | \bar{\mathcal{F}}_{s,t}].$$

We call $(\langle M \rangle_z)$ the *quadratic variation* of (M_z) .

A particular example of strong martingale is the H -valued Wiener process with two parameters which is defined as a process $(w_z)_{z \in I}$ with the following properties :

(k₁) (w_z) vanishes on I_0 and has continuous paths.

(k₂) $E(w_z) = 0$ and $w_z \in L_2(\Omega, \mathcal{F}_z, P, H)$ for each z .

(k₃) $\text{Cov}(\square_{s',t'} w_{s,t}) = (s' - s)(t' - t)W$, where W is a positive, trace class operator on H .

(k₄) the increments over disjoint rectangles are independent.

The quadratic variation of (w_z) is $(\mu((0, z]) \text{Tr } W)$, where $\text{Tr } W$ is the trace of W .

We shall note $\mathcal{P}$ the σ -algebra on $I \times \Omega$ generated by the sets $(z, z'] \times A, A \in \mathcal{F}_z$. The elements of $\mathcal{P}$ are called predictable sets and a H -valued process is $\mathcal{F}_z$ -predictable if it is measurable relative to $(\mathcal{P}, \mathcal{B}_H)$, $\mathcal{B}_H$ being the σ -algebra of Borel sets from H .

Let $S(H)$ be the vector space defined by

$$S(H) = \{\varphi(z, \omega) : I \times \Omega \rightarrow \mathcal{L}(H); \text{ there exists a partition } (z_{i,j})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \text{ of } I \text{ such that } \varphi = \varphi_{i,j} \text{ on } (z_{i,j}, z_{i+1,j+1}], \text{ with } \varphi_{i,j} \text{ } \mathcal{F}_{z_{i,j}}\text{-measurable and bounded}\}.$$

For $\varphi \in S(H)$ we define the stochastic integral $\int \varphi dM$ by

$$\int \varphi dM = \sum_{i=1}^n \sum_{j=1}^q \varphi_{i,j}(\square_{s_{i+1}, t_{j+1}} M_{s_i, t_j})$$

The stochastic integral $\int_J \varphi dM$ is a linear mapping defined on $S(H)$ with values in $L_2(\Omega, \mathcal{F}, P, H)$ which satisfies

$$E\left(\left|\int_J \varphi dM\right|^2\right) \leq E\left(\int_J \|\varphi(u)\|^2 d\langle M \rangle_u\right)$$

Let $\bar{S}(H)$ be the completion of $S(H)$ with respect to the norm $E\left(\int_J \|\varphi\|^2 d\langle M \rangle\right)$.

The above inequality shows that the mapping $\varphi \rightarrow \int_J \varphi dM$ defined on $S(H)$ into $L_2(\Omega, \mathcal{F}, P, H)$ is uniformly continuous, so that there is an unique extension of this mapping in a linear continuous mapping from $\bar{S}(H)$ to $L_2(\Omega, \mathcal{F}, P, H)$ which also satisfies the same inequality.

The image of $\varphi \in \bar{S}(H)$ by this mapping will be noted by $\int_J \varphi dM$ and will be called the stochastic integral of φ with respect to M .

Remark. One can prove that if φ is predictable and satisfies

$$E\left(\int_J \|\varphi\|_2^2 d\langle M \rangle\right) < \infty$$

then $\varphi \in \bar{S}(H)$.

For each z we shall define

$$\int_{(0, z]} \varphi dM = \int_J \lambda_{(0, z]} \varphi dM$$

The process $\left(\int_{(0, z]} \varphi dM\right)_z$ has a continuous version (we denote it in the same way) that is a strong martingale.

Also we note the validity of the following Doob inequality :

$$E\left(\sup_{u \leq z} \left|\int_{(0, u]} \varphi dM\right|^2\right) \leq 16E\left(\int_{(0, z]} \|\varphi\|_2^2 d\langle M \rangle\right)$$

2. Stochastic differential equations.

We denote

$$D(I, H) = \{f: I \rightarrow H; f \text{ is right continuous and with left limits}\};$$

$\mathcal{R}_z$ = the borelian field generated by the sets of the form

$$\{g \in D(I, H); (g(z_1), \dots, g(z_k)) \in A^k, k \geq 1, z \in I, A^k \in \mathcal{B}_{H^k}\}$$

By using the monotone class theorem is not difficult to prove the following lemma.

Lemma 2.1. Let $\varphi(z, g, \omega): I \times D(I, H) \times \Omega \rightarrow B$, B a Banach space, be a $\mathcal{R}_z \otimes \mathcal{F}_z$ - predictable process and (x_z) be an $\mathcal{F}_z$ -adapted right continuous and with left limits process with values in H .

Then $\psi_z(\omega) = \varphi(z, x, \omega)$ is $\mathcal{F}_z$ -predictable and depends only on $(x_u(\omega))_{u < z}$.

Theorem 2.2. Suppose we are given :

(i₁) $(M_z)_{z \in I}$ a continuous H -valued process which is a square integrable strong martingale with quadratic variation $\langle M \rangle$ satisfying

$$\square_{s', t'} \langle M \rangle_{s, t} \leq C_1(s' - s)(t' - t)$$

(i₂) $(V_z)_{z \in I}$ a continuous H -valued adapted process with finite variation

$$A_z = \int_{[0, z]} d|V_u|$$

that satisfies

$$\square_{s', t'} A_{s, t} \leq C_2(s' - s)(t' - t)$$

(i₃) $(\eta_z)_{z \in I}$ an adapted right continuous and with left limits H -valued process such that

$$E(\sup_z |\eta_z|^2) = C_3 < \infty$$

(i₄) $a: I \times D(I, H) \times \Omega \rightarrow \mathcal{L}_2(H)$

$$b: I \times D(I, H) \times \Omega \rightarrow \mathcal{L}_2(H)$$

$\mathcal{R}_z \otimes \mathcal{F}_z$ -predictable processes such that :

$$\|a(z, h, \omega) - a(z, g, \omega)\|_2^2 \leq K_1 \sup_{u \leq z} |h(u) - g(u)|^2,$$

$$\|b(z, h, \omega) - b(z, g, \omega)\|_2^2 \leq K_2 \sup_{u \leq z} |h(u) - g(u)|^2,$$

$$\|a(z, h, \omega)\|_2^2 \leq K_3 + K_4 \sup_{u \leq z} |h(u)|^2,$$

$$\|b(z, h, \omega)\|_2^2 \leq K_5 + K_6 \sup_{u \leq z} |h(u)|^2.$$

Then there exists a unique H -valued process (x_z) which is adapted right continuous and with left limits (if η is continuous so is x) and such that

$$(1) \quad x_z = \eta_z + \int_{(0,z]} a(u, x) dM_u + \int_{(0,z]} b(u, x) dV_u \text{ a.s.}$$

Moreover the following inequality holds :

$$(2) \quad E(\sup_{u \leq z} |x_u|^2) \leq [3E(\sup_{u \leq z} |\eta_u|^2) + (48C_1K_3 + 3C_2^2K_5)\mu((0, z])].$$

$$\exp\{(48C_1K_4 + 3C_2^2K_6)\mu((0, z])\}$$

We call (x_z) a strong solution of the stochastic equation (1).

Proof. Since it is used the well known method of successive approximations we only sketch the proof of the theorem.

We define by recurrence the following sequence $(x_z^n)_{z \in I}$ of processes :

$$x_z^0 = \eta_z$$

$$x_z^{n+1} = \eta_z + \int_{(0,z]} a(u, x^n) dM_u + \int_{(0,z]} b(u, x^n) dV_u$$

If we put

$$\beta_z^n = \sup_{u \leq z} |x_u^{n+1} - x_u^n|^2$$

then we obtainn

$$E(\beta_{s,t}^n) \leq \frac{C^n}{(n!)^2} s^n t^n$$

where C is a constant depending only on C_i, K_i .

Then by the Borel-Cantelli lemma we deduce that x_z^n converges a.s. (uniformly on I) to a process x_z which is a solution of (1).

Let $(x_z), (y_z)$ be two strong solutions of (1). Without loss of generality we may assume that $\max_z \{\sup(|x_z|, |y_z|)\} = \bar{K} < \infty$ (eventually we pass to stopped processes).

Then we get

$$E(\sup_{u \leq z} |x_u - y_u|^2) \leq \bar{C} \int_{(0,z]} E(\sup_{v \leq u} |x_v - y_v|^2) du$$

with $\bar{C}$ depending only on K_1, K_2, C_1, C_2 .

This implies that $E(\sup_{u \leq z} |x_u - y_u|^2) = 0$, hence $x_z = y_z$ a.s. and this proves the unicity.

Next we have

$$\begin{aligned}
 f(z) = E(\sup_{u \leq z} |x_u|^2) &\leq 3E(\sup_{u \leq z} |\eta_u|^2) + 48C_1 \int_{(0, z]} E(\|a(u, x)\|_2^2) du + \\
 &+ 3C_2^2 \int_{(0, z]} E(\|b(u, x)\|_2^2) du \leq 3E(\sup_{u \leq z} |\eta_u|^2) + (48C_1K_3 + 3C_2^2K_5)\mu((0, z]) + \\
 &+ (48C_1K_4 + 3C_2^2K_6) \int_{(0, z]} f(u) du
 \end{aligned}$$

where from we get (2).

Next for simplicity we denote

$$t_i = \frac{i}{2^n}, D_{i,j} = [t_i, t_{i+1}] \times [t_j, t_{j+1}], i, j = 0, \dots, 2^n - 1$$

As an application of (2) we obtain:

Proposition 2.3. Let M, V, η, a, b , be as in theorem 2.2 and moreover assume that

$$(3) \quad |\eta_z - \eta_{z'}|^2 \leq C_4 |z - z'|^2$$

$$(4) \quad 48C_1K_4 + 3C_2^2K_6 \leq 2^n$$

Then we have

$$(5) \quad E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^2) \leq d_n = \alpha_1 2^{-4n} + \alpha_2 2^{-2n} + \alpha_3 2^{-n}$$

where

$$\begin{aligned}
 \alpha_1 &= 7C_2^2K_5 + 38C_2^2K_6(3C_3 + 48C_1K_3 + 3C_2^2K_5) \exp(48C_1K_4 + 3C_2^2K_6) \\
 \alpha_2 &= 14C_4 + 14C_2^2K_5 + 112C_1K_3 + (3C_3 + 48C_1K_3 + 3C_2^2K_5)(448C_1K_4 + \\
 &\quad + 28C_2K_6) \exp(48C_1K_4 + 3C_2^2K_6) \\
 \alpha_3 &= 56C_1K_3 + 112C_1K_4(3C_3 + 48C_1K_3 + 3C_2^2K_5) \exp(48C_1K_4 + 3C_2^2K_6)
 \end{aligned}$$

3. Approximations theorems

In this paragraph we keep the earlier notations and the hypothesis (i₁) – (i₄), (3), (4) are supposed true.

The following inequality is extremely elementary but is often useful. It should be viewed as the Gronwall inequality in the discrete case

Lemma 3.1. Let $\alpha, \beta > 0$ and $f: \{(t_i, t_j); i, j = 0, \dots, 2^n - 1\} \rightarrow \mathbb{R}$ be such that $f_{0,t} = f_{s,0} = 0$; $\square_{s',t'} f_{s,t} \geq 0$; $2^{2n} f(D_{i,j}) \leq \alpha + \beta f_{t_i, t_j}$

Then $f_{s,t} \leq \alpha$ st. exp (βst)

Proof. Induction.

Theorem 3.2. For each n let $(x_{s,t}^n)$ be the process defined by:

$$x_{t_i,0}^n = \eta_{t_i,0}; \quad x_{0,t_j}^n = \eta_{0,t_j}$$

$$x_{t_i, t_j}^n = \eta_{t_i, t_j} + \int_{(0, t_i] \times (0, t_j]} a(u, x^n) dM_u + \int_{(0, t_i] \times (0, t_j]} b(u, x^n) dV_u$$

$$x_{s,t}^n = x_{t_i, t_j}^n \text{ if } t_i \leq s < t_{i+1}, \quad t_j \leq t < t_{j+1}$$

Then we have

$$\mathbb{E} \left(\sup_{\substack{u \leq t_i \\ v \leq t_j}} |x_{u,v} - x_{u,v}^n|^2 \right) \leq d_{n,i,j} = \beta_1 2^{-4n} + \beta_2 2^{-2n} + (96 C_1 K_1 + 24 C_2^2 K_2) \cdot$$

$$(6) \quad \cdot (d_n + \beta_1 2^{-4n} + \beta_2 2^{-2n}) t_i t_j \exp \{ (96 C_1 K_1 + 24 C_2^2 K_2) t_i t_j \}$$

where

$$\beta_1 = 12 C_2^2 K_5 + 48 C_2^2 K_6 (3 C_3 + 48 C_1 K_3 + 3 C_2^2 K_5) \exp(48 C_1 K_4 + 3 C_2^2 K_6)$$

$$\beta_2 = 6 C_4 + 24 C_2^2 K_5 + 48 C_2^2 K_6 (3 C_3 + 48 C_1 K_3 + 3 C_2^2 K_5) \exp(48 C_1 K_4 + 3 C_2^2 K_6)$$

Proof. We have

$$\begin{aligned} \mathbb{E} \left(\sup_{\substack{u \leq t_i \\ v \leq t_j}} |x_{u,v} - x_{u,v}^n|^2 \right) &\leq 6 C_4 2^{-2n} + 48 C_1 \int_0^{t_i} \int_0^{t_j} \mathbb{E} (\|a(q, x) - a(q, x^n)\|_2^2) dq + \\ &+ 3 \mathbb{E} \left(\sup_{\substack{k \leq i \\ l \leq j}} \sup_{p \in D_{k,l}} \left| \int_0^p b(q, x) dV_q - \int_0^p b(q, x^n) dV_q \right|^2 \right) \leq \beta_1 2^{-4n} + \beta_2 2^{-2n} + \\ &+ 48 C_1 \int_0^{t_i} \int_0^{t_j} \mathbb{E} (\|a(q, x) - a(q, x^n)\|_2^2) dq + 12 C_2^2 \int_0^{t_i} \int_0^{t_j} \mathbb{E} (\|b(q, x) - b(q, x^n)\|_2^2) dq \\ &= \beta_1 2^{-4n} + \beta_2 2^{-2n} + y_{t_i, t_j} \\ y(D_{i,j}) &\leq 48 C_1 2^{-2n} \mathbb{E} \left(\sup_{q \in D_{i,j}} \|a(q, x) - a(q, x^n)\|_2^2 \right) + \\ &+ 12 C_2^2 2^{-2n} \mathbb{E} \left(\sup_{q \in D_{i,j}} \|b(q, x) - b(q, x^n)\|_2^2 \right) \leq (96 C_1 K_1 + 24 C_2^2 K_2) (y_{t_i, t_j} + \\ &+ d_n + \beta_1 2^{-4n} + \beta_2 2^{-2n}) 2^{-2n} \end{aligned}$$

Now the conclusion follows by an application of lemma 3.1.

Theorem 3.3. We add to $(i_1) - (i_4)$, (3), (4) the assumption

$$\begin{aligned} \|a(q, x) - a(q', x)\|_2^2 &\leq L_1 \sup_{q \leq u \leq q'} |x_u - x_q|^2; \|b(q, x) - b(q', x)\|_2^2 \leq \\ &\leq L_2 \sup_{q \leq u \leq q'} |x_u - x_q|^2 \end{aligned}$$

We define for every n :

$$\begin{aligned} x_{t_i, 0}^n &= \eta_{t_i, 0}; \quad x_{0, t_j}^n = \eta_{0, t_j} \\ x_{t_{i+1}, t_{j+1}}^n &= \eta_{t_{i+1}, t_{j+1}} + \sum_{\mu=0}^i \sum_{v=0}^j [a(t_\mu, t_v, x^n)(\square_{t_{\mu+1}, t_{v+1}} M_{t_\mu, t_v}) + \\ &\quad + b(t_\mu, t_v, x^n)(\square_{t_{\mu+1}, t_{v+1}} V_{t_\mu, t_v})] \\ x_{s, t}^n &= x_{t_i, t_j}^n \text{ if } t_i \leq s < t_{i+1}, t_j \leq t < t_{j+1} \end{aligned}$$

Then

$$\begin{aligned} (7) \quad E(\sup_{\substack{u \leq t_i \\ v \leq t_j}} |x_{u, v} - x_{u, v}^n|^2) &\leq \beta_1 2^{-4n} + \beta_2 2^{-2n} + (96 C_1 L_1 + 24 C_2^2 L_2) (d_n + \\ &\quad + \beta_1 2^{-4n} + \beta_2 2^{-2n}) t_i t_j \exp \{(96 C_1 L_1 + 24 C_2^2 L_2) t_i t_j\} \end{aligned}$$

Proof. If we put $\bar{a}(u, x) = a(t_i, t_j, x)$, $\bar{b}(u, x) = b(t_i, t_j, x)$ for $u \in [t_i, t_{i+1}] \times [t_j, t_{j+1}]$ then we have

$$x_{t_i, t_j}^n = \eta_{t_i, t_j} + \int_{(0, t_i] \times (0, t_j]} \bar{a}(u, x^n) dM_u + \int_{(0, t_i] \times (0, t_j]} \bar{b}(u, x^n) dV_u$$

Next the proof continues as in theorem 3.2.

Theorem 3.4. Let M, V, η that satisfy $(i_1) - (i_3)$ (with η continuous) and let $a, b: I \times H \times \Omega \rightarrow \mathcal{L}_2(H)$ be $\mathcal{F}_z$ -adapted and continuous processes such there exists the constants $k_i, \bar{k}_i, i = 1, \dots, 6$, with properties:

$$\begin{aligned} \|a(z, h, \omega) - a(z, g, \omega)\|_2^2 &\leq k_1 |h - g|^2; \|b(z, h, \omega) - b(z, g, \omega)\|_2^2 \leq \\ &\leq \bar{k}_1 |h - g|^2 \end{aligned}$$

$$\|a(z, h, \omega)\|_2^2 \leq k_2; \|b(z, h, \omega)\|_2^2 \leq \bar{k}_2$$

$$\left\| \frac{\partial a}{\partial x}(z, h, \omega) \right\| \leq k_3; \quad \left\| \frac{\partial b}{\partial x}(z, h, \omega) \right\| \leq \bar{k}_3$$

$$\left\| \frac{\partial^2 a}{\partial x^2}(z, h, \omega) \right\| \leq k_4; \quad \left\| \frac{\partial^2 b}{\partial x^2}(z, h, \omega) \right\| \leq \bar{k}_4$$

$$\left\| \frac{\partial a}{\partial s}(z, h, \omega) \right\| \leq k_5; \quad \left\| \frac{\partial b}{\partial s}(z, h, \omega) \right\| \leq \bar{k}_5$$

$$\left\| \frac{\partial a}{\partial t}(z, h, \omega) \right\| \leq k_6; \quad \left\| \frac{\partial b}{\partial t}(z, h, \omega) \right\| \leq \bar{k}_6$$

For each n let (x_s^n) be the process defined by

$$\begin{aligned} x_{s,t}^n = & \eta_{s,t} + \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} [a(t_\mu, t_\nu, x_{t_\mu, t_\nu}^n)(\square_{t_{\mu+1}, t_{\nu+1}} M_{t_\mu, t_\nu}) + \\ & + b(t_\mu, t_\nu, x_{t_\mu, t_\nu}^n)(\square_{t_{\mu+1}, t_{\nu+1}} V_{t_\mu, t_\nu}) + \int_{t_\mu}^{t_{\mu+1}} \int_{t_\nu}^{t_{\nu+1}} \left(a \cdot \frac{\partial a}{\partial x} \right) (t_\mu, t_\nu, x_{t_\mu, t_\nu}^n) \\ & (\square_{t_\mu, t_\nu} M_{u,v}) dM_{u,v} + \int_{t_\mu}^{t_{\mu+1}} \int_{t_\nu}^{t_{\nu+1}} \left(a \cdot \frac{\partial b}{\partial x} \right) (t_\mu, t_\nu, x_{t_\mu, t_\nu}^n) (\square_{t_\mu, t_\nu} M_{u,v}) dV_{u,v} + \\ & + a(t_i, t_j, x_{t_i, t_j}^n)(\square_{s,t} M_{t_i, t_j}) + b(t_i, t_j, x_{t_i, t_j}^n)(\square_{s,t} V_{t_i, t_j}) + \\ & + \int_{t_i}^s \int_{t_j}^t \left(a \cdot \frac{\partial b}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) (\square_{u_1, u_2} M_{t_i, t_j}) dM_{u_1, u_2} + \\ & + \int_{t_j}^s \int_{t_i}^t \left(a \cdot \frac{\partial b}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) (\square_{u_1, u_2} M_{t_i, t_j}) dV_{u_1, u_2} \end{aligned}$$

if $(s, t) \in [t_i, t_{i+1}) \times [t_j, t_{j+1})$.

Then

$$\begin{aligned} (8) \quad E(\sup_{\substack{s \leq t_i \\ t \leq t_j}} |x_{s,t} - x_{s,t}^n|^2) \leq & 6[(\bar{\alpha}_1 + \bar{\beta}_1) + (\bar{\alpha}_2 + \bar{\beta}_2) 2^{-n} + \\ & + (\bar{\alpha}_3 + \bar{\beta}_3) 2^{-2n} + (\bar{\alpha}_4 + \bar{\beta}_4) 2^{-4n}] t_i t_j \exp \{ (96 C_1 k_1 + 6 C_2^2 \bar{k}_1) + \\ & + 12.16^2 C_1^2 (k_2 k_4^2 + k_3^2 k_1) + 192 C_1 C_2^2 (k_2 \bar{k}_4^2 + \bar{k}_3^2 k_1) \} \end{aligned}$$

where $\bar{\alpha}_i, \bar{\beta}_i$ are given at the end of the proof.

Proof. Let us define $\bar{a}(u, x^n) = a(t_i, t_j, x_{t_i, t_j}^n)$ if $u \in [t_i, t_{i+1}) \times [t_j, t_{j+1})$ and similar for partial derivatives.

Also we put $\bar{M}_u = \square_u M_{s, t_j}$ for $u \in [t_i, t_{i+1}) \times [t_j, t_{j+1})$

Then we have

$$\begin{aligned} x_z^n = \eta_z + \int_{(0, z]} \bar{a}(u, x^n) dM_u + \int_{(0, z]} \bar{b}(u, x^n) dV_u + \int_{(0, z]} \left(\bar{a} \cdot \frac{\partial \bar{a}}{\partial x} \right)(u, x^n) \bar{M}_u dM_u \\ + \int_{(0, z]} \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right)(u, x^n) \bar{M}_u dV_u \end{aligned}$$

It follows that

$$E\left(\sup_{\substack{s \leq t_i \\ t \leq t_j}} |x_{s, t} - x_{s, t}^n|^2\right) \leq 6y_{t_i, t_j} = 6 \sum_{k=1}^6 J_{t_i, t_j}^k$$

$$y(D_{i, j}) = \sum_{k=1}^6 J^k(D_{i, j})$$

Next we obtain

$$J^3(D_{i, j}) = 16C_1 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} E(|\bar{a}(u, x^n) - \bar{a}(u, x)|^2) du =$$

$$= 16C_1 2^{-2n} E(|a(t_i, t_j, x_{t_i, t_j}^n) - a(t_i, t_j, x_{t_i, t_j}^n)|^2) \leq$$

$$\leq 16C_1 2^{-2n} k_1 \cdot 6y_{t_i, t_j} = 96C_1 k_1 2^{-2n} y_{t_i, t_j}$$

$$J^4(D_{i, j}) = C_2^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} E(|\bar{b}(u, x) - \bar{b}(u, x^n)|^2) du \leq 6C_2^2 k_1 2^{-2n} y_{t_i, t_j}$$

$$J^5(D_{i, j}) = 16C_1 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} E\left(\left|\left[\left(\bar{a} \cdot \frac{\partial \bar{a}}{\partial x}\right)(u, x) - \left(\bar{a} \cdot \frac{\partial \bar{a}}{\partial x}\right)(u, x^n)\right] \bar{M}_u\right|^2\right) du \leq$$

$$16C_1 2^{-2n} E\left(\sup_{u \in D_{i, j}} \left|\int_{t_i}^{u_1} \int_{t_j}^{u_2} \left[\left(a \cdot \frac{\partial a}{\partial x}\right)(t_i, t_j, x_{t_i, t_j}^n) - \right.\right.\right.$$

$$\left.\left. - \left(a \cdot \frac{\partial a}{\partial x}\right)(t_i, t_j, x_{t_i, t_j}^n)\right] dM_{u_1, u_2}\right|^2\right) \leq 16^2 C_1^2 2^{-2n} E\left(\left|\left(a \cdot \frac{\partial a}{\partial x}\right)(t_i, t_j, x_{t_i, t_j}^n) - \right.\right.$$

$$\begin{aligned}
& - \left(a \cdot \frac{\partial a}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) |^2 \leq 2 \cdot 16^2 C_1^2 2^{-2n} \{ E(|a(t_i, t_j, x_{t_i, t_j}^n) \cdot \\
& \cdot \left[\frac{\partial a}{\partial x} (t_i, t_j, x_{t_i, t_j}^n) - \frac{\partial a}{\partial x} (t_i, t_j, x_{t_i, t_j}^n) \right]^2) + E(|[a(t_i, t_j, x_{t_i, t_j}^n) - \\
& - a(t_i, t_j, x_{t_i, t_j}^n)] \cdot \frac{\partial a}{\partial x} (t_i, t_j, x_{t_i, t_j}^n) |^2) \} \leq 12 \cdot 16^2 C_1^2 2^{-2n} (k_2 k_4^2 + k_3^2 k_1) y_{t_i, t_j} \\
J^6(D_{i,j}) & \leq C_2^2 2^{-2n} \cdot 16 C_1 E \left(\left| \left(a \cdot \frac{\partial b}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) - \left(a \cdot \frac{\partial b}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) \right|^2 \right) \\
& \leq 192 C_1 C_2^2 2^{-2n} (k_2 k_4^2 + k_3^2 k_1) y_{t_i, t_j} \\
J^1(D_{i,j}) & \leq 16 C_1 2^{-2n} E(\sup_{u \in D_{i,j}} |a(u, x_u) - a(t_i, t_j, x_{t_i, t_j}^n) - \\
& - \left(a \cdot \frac{\partial a}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) \bar{M}_u|^2)
\end{aligned}$$

By using the Taylor formula we obtain

$$\begin{aligned}
a(u, x_u) - a(t_i, t_j, x_{t_i, t_j}^n) &= a(u, x_u) - a(t_i, t_j, x_u) + a(t_i, t_j, x_u) - \\
& - a(t_i, t_j, x_{t_i, t_j}^n) = \frac{\partial a}{\partial s}(\cdot)(u_2 - t_i) + \frac{\partial a}{\partial t}(\cdot)(u_2 - t_j) + \\
& + \frac{\partial a}{\partial x}(t_i, t_j, x_{t_i, t_j}^n)(x_u - x_{t_i, t_j}^n) + \frac{1}{2} \frac{\partial^2 a}{\partial x^2}(\cdot)(x_u - x_{t_i, t_j}^n)^2
\end{aligned}$$

where from

$$\begin{aligned}
& a(u, x_u) - a(t_i, t_j, x_{t_i, t_j}^n) - \left(a \cdot \frac{\partial a}{\partial x} \right) (t_i, t_j, x_{t_i, t_j}^n) \square_u \bar{M}_{t_i, t_j} = \\
& = \frac{\partial a}{\partial s}(\cdot)(u_1 - t_i) + \frac{\partial a}{\partial t}(\cdot)(u_2 - t_j) + \frac{1}{2} \frac{\partial^2 a}{\partial x^2}(\cdot)(x_u - x_{t_i, t_j}^n)^2 + \\
& + \frac{\partial a}{\partial x}(t_i, t_j, x_{t_i, t_j}^n) \cdot
\end{aligned}$$

$$\left\{ \eta_u - \eta_{t_i, t_j} + \int_{t_i}^{u_1} \int_{t_j}^{u_2} [a(q, x_q) - a(t_i, t_j, x_{t_i, t_j})] dM_q + \int_{t_i}^{u_1} \int_{t_j}^{u_2} b(q, x_q) dV_q + \right. \\ \left. + \int_0^{t_i} \int_{t_j}^{u_2} + \int_{t_i}^{u_1} \int_0^{t_j} (adM + bdV) \right\}$$

It follows that

$$J^1(D_{i,j}) \leq 10.16 C_1 2^{-2n} \left\{ E \left(\sup \left| \frac{\partial a}{\partial s} (u_1 - t_i) \right|^2 \right) + E \left(\sup \left| \frac{\partial a}{\partial t} (u_2 - t_j) \right|^2 \right) + \right. \\ \left. + \frac{1}{4} E \left(\sup \left| \frac{\partial^2 a}{\partial x^2} (x_u - x_{t_i, t_j}) \right|^2 \right) + E \left(\sup \left| \frac{\partial a}{\partial x} (\eta_u - \eta_{t_i, t_j}) \right|^2 \right) + \right. \\ \left. + E \left(\sup \left| \frac{\partial a}{\partial x} \int_{t_i}^{u_1} \int_{t_j}^{u_2} [a(q, x_q) - a(t_i, t_j, x_{t_i, t_j})] dM_q \right|^2 \right) + \right. \\ \left. + E \left(\sup \left| \frac{\partial a}{\partial x} \int_{t_i}^{u_1} \int_{t_j}^{u_2} b dV \right|^2 \right) + E \left(\sup \left| \frac{\partial a}{\partial x} \int_0^{t_i} \int_{t_j}^{u_2} a dM \right|^2 \right) + \right. \\ \left. + E \left(\sup \left| \frac{\partial a}{\partial x} \int_0^{t_i} \int_{t_j}^{u_2} a dM \right|^2 \right) + E \left(\sup \left| \frac{\partial a}{\partial x} \int_0^{t_i} \int_{t_j}^{u_2} b dV \right|^2 \right) \right. \\ \left. + E \left(\sup \left| \frac{\partial a}{\partial x} \int_{t_i}^{u_1} \int_0^{t_j} b dV \right|^2 \right) \leq 160 C_1 2^{-2n} \left\{ k_5^2 2^{-2n} + k_6^2 2^{-2n} + \right. \\ \left. + \frac{1}{4} k_4^2 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^4) + k_3^2 2^{-2n+1} + k_3^2 \cdot 16 C_1 \int_{D_{i,j}} E(\|a(q, x_q) - \right. \\ \left. - a(t_i, t_j, x_{t_i, t_j})\|_2^2 dq) + 2^{-2n} k_7^2 C_2^2 \int_{D_{i,j}} E(\|b(q, x_q)\|_2^2) dq + 16 C_1 k_3^2 \left(\int_0^{t_i} \int_{t_j}^{u_2} + \right. \right. \\ \left. \left. + \int_{t_i}^{u_1} \int_0^{t_j} E(\|a(q, x_q)\|_2^2) dq \right) + 2^{-n} C_2^2 k_3^2 (t_i + t_j) \right\}$$

Now taking into account that

$$\begin{aligned} \|a(q, x_q) - a(t_i, t_j, x_{t_i, t_j})\|_2^2 &\leq 2(\|a(q, x_q) - a(q, x_{t_i, t_j})\|_2^2 + \|a(q, x_{t_i, t_j}) - \\ &- a(t_i, t_j, x_{t_i, t_j})\|_2^2) \leq 2\left(k_1 \sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^2 + 2|(u_1 - t_i) \frac{\partial a}{\partial s}|^2 + \right. \\ &\quad \left. + 2|(u_2 - t_j) \frac{\partial a}{\partial t}|^2\right) \end{aligned}$$

we get

$$\begin{aligned} J^1(D_{i,j}) &\leq 160 C_1 2^{-2n} \{2^{-2n} k_5^2 + 2^{-2n} k_6^2 + \frac{1}{4} k_4^2 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^2) + \\ &+ 2^{-2n+1} k_3^2 + 16 C_1 k_3^2 2^{-2n} [2 k_1 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^2) + 4 k^2 2^{-2n} + 4 k_6^2 2^{-2n}] \cdot \\ &+ 2^{-4n} k_3^2 C_2^2 \bar{k}_2 + 16 C_1 k_2 k_3^2 (t_i + t_j) 2^{-n} + k_3^2 C_2^2 (t_i + t_j) 2^{-n}\} \end{aligned}$$

so that

$$J^1(D_{i,j}) \leq \bar{\alpha}_1 2^{-2n} + \bar{\alpha}_2 2^{-3n} + \bar{\alpha}_3 2^{-4n} + \bar{\alpha}_4 2^{-6n}$$

where

$$\bar{\alpha}_1 = 40 C_1 k_4^2 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^4)$$

$$\bar{\alpha}_2 = 160 (32 k_2 k_3^2 C_1 + 2 C_2^2 k_3^2)$$

$$\bar{\alpha}_3 = 160 [k_5^2 + k_6^2 + 2 k_3^2 + 32 C_1 k_1 k_3^2 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^2) + C_2^2 k_3^2 \bar{k}_2]$$

$$\bar{\alpha}_4 = 160 (64 C_1 k_3^2 k_5^2 + 64 C_1 k_6^2 k_3^2 + k_3^2 C_2^2 \bar{k}_2)$$

Similar we obtain

$$J^2(D_{i,j}) \leq \beta_1 2^{-2n} + \beta_2 2^{-3n} + \beta_3 2^{-4n} + \beta_4 2^{-6n}$$

where

$$\bar{\beta}_1 = \frac{10}{4} C_2^2 \bar{k}_4^2 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^4)$$

$$\bar{\beta}_2 = 10 C_2^2 (16 k_2 \bar{k}_3^2 C_1 + 16 k_2 k_3^2 C_1 + \bar{k}_3^2 C_2^2 + k_3^2 C_2^2)$$

$$\bar{\beta}_3 = 10 C_2^2 [k_5^2 + \bar{k}_6^2 + 2 \bar{k}_3^2 + 32 C_1 \bar{k}_3^2 k_1 E(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^2) + \bar{k}_3^2 C_2^2 \bar{k}_2]$$

$$\bar{\beta}_4 = 10 C_2^2 (\bar{k}_3^2 C_2^2 \bar{k}_2 + 64 C_1 k_5^2 \bar{k}_3^2 + 64 C_1 \bar{k}_3^2 k_6^2)$$

Now the inequality (8) follows easily.

Proposition 3.5. Under the hypothesis of theorem 3.4. and moreover if M_z is a H -valued Wiener process the following inequality holds :

$$(9) \quad E \left(\sup_{u \in D_{i,j}} |x_u - x_{t_i, t_j}|^4 \right) \leq 7^3 C_2^4 \tilde{k}_2^2 2^{-8n} + 7^3 \left[C_2^4 \tilde{k}_2^2 (t_i^4 + t_j^4) + 36 \left(\frac{3}{4} \right)^4 k_2^2 \cdot \right. \\ \left. \cdot (1 + \|W\|^2)^2 + 4C_4^2 \right] 2^{-4n} + 36 \cdot 7^3 \left(\frac{4}{3} \right)^4 k_2^2 (1 + \|W\|^2)^2 (t_i^2 + t_j^2) 2^{-2n}$$

Proof. From the Hölder inequality we have

$$\left| \int_s^u \int_t^v b(q, x_q) dV_q \right|^4 \leq \left(\int_s^u \int_t^v dA \right)^3 \left(\int_s^u \int_t^v \|b(q, x_q)\|_2^4 dA \right) \leq \\ \leq C_2^4 \tilde{k}_2^2 (u - s)^4 (v - t)^4$$

Also from the Doob inequality we get

$$E \left(\sup_{\substack{s \leq u \leq s' \\ t \leq v \leq t'}} \left| \int_s^u \int_t^v a(q, x_q) dM_q \right|^4 \right) \leq \left(\frac{4}{3} \right)^4 M \left(\left| \int_s^{s'} \int_t^{t'} a(q, x_q) dM_q \right|^4 \right)$$

It is known that for $h \in H$ the process $\left\langle \int_s^u \int_t^v a(q, x_q) dM_q, h \right\rangle$ is a strong martingale with $\int_s^u \int_t^v \langle (aWa^*)(q, x_q)h, h \rangle dq$ as characteristic.

Then $(y_u)_{u \geq s} = \left(\int_s^u \int_t^v \langle a(q, x_q) dM_q, h \rangle \right)_{u \geq s}$ is a martingale with

$$\left(\int_s^u \int_t^v \langle (aWa^*)(q, x_q)h, h \rangle dq \right)_{u \geq s}$$

as increasing process.

By applying the Ito formula to the function $f(x) = |x|^4$ (see [6], theorem 1.4) and taking into account that

$$f''(x_0)(y, z) = 2|x_0|^2 \langle y, z \rangle + 4\langle x_0, y \rangle \langle x_0, z \rangle \leq 6|x_0|^2 |y| |z|$$

$$\begin{aligned}
\text{hence } \text{Tr} [(aW a^*) (q, x_q) f''(x)] &= \sum_{n=1}^{\infty} \langle f''(x_q) a(q, x_q) e_n, (aW) (q, x_q) e_n \rangle \leq \\
&\leq 6 \sum_{n=1}^{\infty} |x|^2 \| (aW) (q, x_q) e_n \| a(q, x_q) e_n \leq 12 |x|^2 \sum_{n=1}^{\infty} (\| (aW) (q, x_q) e_n \|^2 + \\
&+ \| a(q, x_q) e_n \|^2) = 12 |x|^2 (\| (aW) (q, x_q) \|^2_2 + \| a(q, x) \|^2_2) \leq \\
&12 |x|^2 k_2 (1 + \| W \|^2)
\end{aligned}$$

we obtain

$$\begin{aligned}
E(|y_{s'}|) &\leq 6k_2(1 + \| W \|^2) (t' - t) \int_s^{s'} E(|y_u|^2) du \leq \\
&\leq 6k_2(1 + \| W \|^2) (t' - t) \int_s^{s'} [E(|y_u|^4)]^{\frac{1}{2}} du \leq \\
&\leq 6k_2(t' - t) (s' - s) [E(|y_{s'}|^4)]^{\frac{1}{2}} (1 + \| W \|^2)
\end{aligned}$$

where from

$$E(|y_{s'}|^4) \leq 36k_2^2(1 + \| W \|^2)^2 (t' - t)^2 (s' - s)^2$$

Now the inequality (9) follows easily.

Observe that if η is constant then we may change the constant 7 with 6 in (9).

A numerical example. Let us consider the following stochastic differential equation :

$$dx_{s,t} = dw_{s,t} + \frac{1}{4} x_{s,t} ds dt$$

$$x_{s,0} = x_{0,t} = 1$$

where $(w_{s,t})$ is a real brownian motion.

By using the theorem 3.3 with $n = 10$ and the rejection procedure for generating normal variables the values of $u_{s,t} = M(x_{s,t})$ are computed in different points.

On the other hand $u_{s,t}$ satisfies the hyperbolic equation

$$\frac{\partial^2 u}{\partial s \partial t} = \frac{1}{4} u$$

$$u_{s,0} = u_{0,t} = 1$$

hence $u_{s,0} = I_0(\sqrt{st})$, with $I_0(t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{t}{2}\right)^{2n}$.

We obtain the following results :

Nr. simulăr	s = t	2 ⁻¹⁰	2 ⁻⁹	2 ⁻⁸	2 ⁻⁷	2 ⁻⁶
50		0,99990539	1,00008624	1,00031765	1,00045890	1,00046158
200		0,99986712	0,99987376	1,00001786	1,00042352	0,99947289
350		0,99989151	0,99996540	1,00026175	1,00001561	0,99998488
$u_{s,t} = I_0(\sqrt{st})$		1,00000023	1,00000095	1,00000381	1,00001525	1,00006103

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SCALAR-TYPE OPERATORS THAT ARE COMMUTATORS OF COMPACT OPERATORS

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In 1972, C. Pearcy [4] posed the following question: „which operators A on Hilbert space can be written as a commutator $A = BC - CB$, where both B and C are compact?” The purpose of this note is to make use of a technique developed in [2] to give a sufficient spectral condition that a scalar-type operator on a Banach space with all multiplicities equal to 1 be a commutator of compact operators. In Hilbert space, we relax the condition, but obtain the same conclusion.

Let X be a non-zero complex Banach space and S a compact scalar-type operator on X [1, Definition 5.14] with all multiplicities equal to 1. It is easy to verify that S can be written in the form

$$S = \sum_i \lambda_i E(\lambda_i)$$

or

$$S = \sum_i \lambda_i F_i(\cdot) x_i$$

where $\{\lambda_i\}_{i \in \mathbb{N}}$ is the sequence of non-zero eigenvalues of S , $E(\lambda_i) = F_i(\cdot)x_i$ is the spectral measure of the single-point set $\{\lambda_i\}$ ($i = 1, 2, \dots$), x_i is a unit vector in subspace $E(\lambda_i)X$ ($i = 1, 2, \dots$) and $F_i(\cdot)$ is an element of X^* ($i = 1, 2, \dots$) with the property

$$F_i(x_j) = \delta_{ij} \quad (i, j = 1, 2, \dots)$$

Since the spectral measure $E(\cdot)$ is bounded, so is the sequence $\{\|F_i\|\}_{i \in \mathbb{N}}$.

Theorem 1. Let S be a compact scalar-type operator on X with all the multiplicities equal to 1. If there is a permutation $\{\lambda_{i_j}\}_{i \in \mathbb{N}}$ of all non-zero eigenvalues of S such that

$$\sum_i |M_i|^{\frac{1}{2}} < \infty$$

($M_i = \lambda_1 + \dots + \lambda_i$, $i = 1, 2, \dots$), then S is a commutator of compact operators.

Proof. Clearly the following series

$$\sum_i M_i F_i(\cdot) x_i; \quad \sum_i M_i F_{i+1}(\cdot) x_{i+1};$$

$$K_1 = \sum_i M_i^{\frac{1}{2}} F_{i+1}(\cdot) x_i; \quad K_2 = \sum_i M_i^{\frac{1}{2}} F_i(\cdot) x_{i+1}$$

are convergent in norm and K_1 and K_2 are compact operators.

Since we have

$$\begin{aligned}
 S &= \sum_i \lambda_i F_i(\cdot) x_i = \sum_i (M_i - M_{i-1}) F_i(\cdot) x_i = \\
 &= \sum_i M_i F_i(\cdot) x_i - \sum_i M_i F_{i+1}(\cdot) x_{i+1} = \\
 &= \sum_i M_i^{\frac{1}{2}} F_{i+1}(\cdot) x_i \sum_i M_i^{\frac{1}{2}} F_i(\cdot) x_{i+1} - \\
 &= \sum_i M_i^{\frac{1}{2}} F_i(\cdot) x_{i+1} \sum_i M_i^{\frac{1}{2}} F_{i+1}(\cdot) x_i = K_1 K_2 - K_2 K_1,
 \end{aligned}$$

the proof is complete.

Let us return to the Hilbert space.

In the sequel as we refer to a permutation $\{\lambda_i\}_{i \in \mathbb{N}}$ of non-zero eigenvalues of a compact operator, each non-zero eigenvalue will be repeated in the permutation as many times as its multiplicity indicates.

Theorem 2. Let T be a compact normal operator on a complex Hilbert space. If there is a permutation $\{\lambda_i\}_{i \in \mathbb{N}}$ of all non-zero eigenvalues of T such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i = 0,$$

then T is a commutator of compact operators.

Proof. We define K_1, K_2 as before, where $F_i(\cdot) = (\cdot, x_i)$ ($i = 1, 2, \dots$) [3, 133].

Remark. Let S be a compact scalar-type operator on a complex Hilbert space. If there exists a permutation $\{\lambda_i\}_{i \in \mathbb{N}}$ of all non-zero eigenvalues of S such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i = 0$$

then S is a commutator of compact operators.

This is true in view of the fact that a (compact) scalar-type operator is similar to a (compact) normal operator [1, Theorem 8.3].

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STABILITY RESULTS FOR RANDOM DECISION SYSTEMS WITH COMPLETE CONNECTIONS

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The concept of a random decision system with complete connections has been introduced in a series of papers [4], [6] — [11]. Some results concerning these systems have been obtained for both the finite horizon case and the infinite horizon case. The study of the random decision systems with complete connections leads us to the study of functional equation (1.1) given below. Some results concerning the stability properties for this equation have been presented in [12]. The purpose of this paper is to give some new stability results for the functional equation (1.1).

1. Introduction

Let us consider three measurable spaces $(X, \mathcal{X})$, $(Y, \mathcal{Y})$, $(W, \mathcal{W})$, a stochastic kernel P defined on $W \times Y \times \mathcal{X}$, and a transformation z of $W \times Y \times X$ into W which is measurable with respect to $\mathcal{W} \times \mathcal{Y} \times \mathcal{X}$. The system $\{(X, \mathcal{X}), (W, \mathcal{W}), (Y, \mathcal{Y}), z, P\}$ is said to be a (homogeneous) random decision system with complete connections.

The concept of a random decision system with complete connections may be regarded as a formalization of the following decision problem. A system S has state space $(X, \mathcal{X})$ and decision space $(Y, \mathcal{Y})$. The space $(W, \mathcal{W})$ is to be regarded as an information space. As soon as the information $w \in W$ about the system S is available and the decision $y \in Y$ is made, we know the distribution $P(w, y, \cdot)$ of the states of the system at the next moment of time. If the new state of the system S is $x \in X$, then $z(w, y, x) \in W$ represents the new („better”) information about the system S which is available at the next moment of time.

The concept of a random decision system with complete connections has been introduced in a series of papers [4], [6] — [11]. Some results concerning these systems have been obtained for both the finite horizon case and the infinite horizon case.

The study of the random decision systems with complete connections leads us to the study of the functional equation

$$(1.1) \quad f(w) = \sup_{y \in Y} (q(w, y) + h(w, y) \int_{\mathcal{X}} P(w, y, dx) f(z(w, y, x))),$$

where q and h are known real valued functions on $W \times Y$.

Some results concerning the stability properties for the functional equation (1.1) may be found in [12]. The purpose of this paper is to give some more stability properties for this functional equation. The main theorems given below show large are the changes in the solution f entailed by changes in the transformation z .

2. Preliminary Results

In the following discussion we shall assume that W is a compact metric space with metric ρ and with Borel sets $\mathcal{W}$. We denote by $C(W)$ the Banach space of bounded continuous real valued functions on W under the norm

$$(2.1) \quad |f| = \sup_{w \in W} |f(w)|.$$

We denote by $B(W, \mathcal{W})$ the Banach space of bounded $\mathcal{W}$ -measurable real valued functions on W under the supremum norm (2.1). Finally, we denote by $CL(W)$ the class of all real valued functions on W which satisfy the conditions

$$(2.2) \quad m(f) = \sup_{\substack{w', w'' \in W \\ w' \neq w''}} \frac{|f(w') - f(w'')|}{\rho(w', w'')} < +\infty.$$

It may be shown that $CL(W) \subset C(W)$ and that $CL(W)$ is a Banach space with the norm

$$(2.3) \quad \|f\| = m(f) + |f|.$$

The proofs of the following lemmas may be found in [11]; in a slightly different manner these preliminary results are also stated and proved in [8].

Lemma 1. Let φ be a real valued function on $W \times Y$. If there exists a real number A so that

$$(2.4) \quad m(\varphi(\cdot, y)) \leq A$$

for any $y \in Y$, then it follows that the function

$$(2.5) \quad f_0(w) = \sup_{y \in Y} \varphi(w, y)$$

belongs to the Banach space $CL(W)$.

Lemma 2. Let $f_1, f_2 \in B(W, W)$, and let φ_1, φ_2 and h be bounded real valued functions on $W \times Y$. If

$$(2.6) \quad F_i(w) = \sup_{y \in Y} (\varphi_i(w, y) + h(w, x) \int_X P(w, y, dx) f_i(z(w, y, x)))$$

exists and is finite for $i = 1, 2$, then

$$(2.7) \quad |F_1(w) - F_2(w)| \leq \sup_{y \in Y} (|\varphi_1(w, y) - \varphi_2(w, y)| + |h(w, y)| \int_X P(w, y, dx) |f_1(z(w, y, x)) - f_2(z(w, y, x))|).$$

Lemma 3. Let $\{(X, \mathcal{X}), (W, \mathcal{W}), (Y, \mathcal{Y}), z, P\}$ be a random decision system with complete connections and let us assume that the following conditions hold:

- (i) *W is a compact metric space with metric ρ and Borel sets $\mathcal{W}$;*
- (ii) *There exists a real number a so that*

$$(2.8) \quad m(P(\cdot, y, A)) \leq a$$

for any $y \in Y$ and any $A \in \mathcal{X}$;

- (iii) *There exists a real number b so that*

$$(2.9) \quad m(h(\cdot, y)) \leq b$$

for any $y \in Y$;

- (iv) *We have*

$$(2.10) \quad |h(w, y)| \leq 1$$

for any $w \in W$ and any $y \in Y$;

- (v) *If we denote*

$$(2.11) \quad \mu(z(\cdot, y, x)) = \sup_{\substack{w', w'' \in W \\ w' \neq w''}} \frac{\rho(z(w', y, x), z(w'', y, x))}{\rho(w', w'')}$$

then we have

$$(2.12) \quad \mu(z(\cdot, y, x)) \leq 1$$

for any $y \in Y$ and $x \in X$.

Under these assumptions, for any $f \in CL(W)$, the function on W defined by

$$(2.13) \quad f^*(w) = \sup_{y \in Y} (h(w, y) \int_{\tilde{X}} P(w, y, dx) f(z(w, y, x)))$$

belongs to $CL(W)$ too. Moreover, we have

$$(2.14) \quad \|f^*\| \leq m(f) + (2a + b + 1)\|f\| \leq (2a + b + 1)\|f\|.$$

Let us introduce the following new assumptions concerning the functional equation (1.1):

- (vi) *There exists a real number B so that*

$$(2.15) \quad m(q(\cdot, y)) \leq B$$

for any $y \in Y$;

- (vii) *There exists $w_0 \in W$ so that*

$$(2.16) \quad q(w_0, y) = 0$$

and

$$(2.17) \quad z(w_0, y, x) = w_1$$

for any $y \in Y$ and $x \in X$;

(viii) There exists $\alpha \in (0,1)$ so that*

$$(2.18) \quad \rho(z(w', y, x), z(w'', y, x)) \leq \alpha \rho(w', w'')$$

for any $w', w'' \in W$, any $y \in Y$ and any $x \in X$;

(ix) If we denote

$$(2.19) \quad W(c) = \{w \in W \mid \rho(w_0, w) \leq c\}$$

and

$$(2.20) \quad v(c) = \sup_{w \in W(c)} \sup_{y \in Y} |q(w, y)|$$

for any real number $c \geq 0$, then

$$(2.21) \quad \sum_{n=0}^{\infty} v(\alpha^n c) < +\infty.$$

If conditions (i) — (ix) hold, the functional equation (1.1) is said to be of type I. The following result is proved in [11].

Theorem 2.1. If the functional equation (1.1) is of type I, then there exists a unique solution $f \in C(W)$ to this equation having the property that $f(w_0) = 0$. Moreover, the sequence of functions (f_n) defined iteratively by

$$(2.22) \quad f_n(w) = \sup_{y \in Y} (q(w, y) + h(w, y) \int_X P(w, y, dx) f_{n-1}(z(w, y, x)))$$

for $n \geq 1$, where $f_0 = 0$, is uniformly convergent to the unique solution f of the functional equation (1.1).

Now, suppose that, instead of conditions (vi) — (ix) given above, we consider the following conditions:

(vi') There exists $\alpha \in (0,1)$ so that

$$(2.23) \quad |h(w, y)| \leq \alpha$$

for any $w \in W$ and any $y \in Y$;

(vii) There exists a real number B so that

$$(2.24) \quad m(q(\cdot, y)) \leq B$$

for any $y \in Y$.

If conditions (i) — (iv) and (vi'), (vii') hold, the functional equation (1.1) is said to be of type II. The following result is proved in [11] too.

Theorem 2.2. If the functional equation (1.1) is of type II, then there exists a unique solution $f \in C(W)$ to this equation. Moreover, the sequence of functions (f_n) defined iteratively by (2.22) is uniformly convergent to the unique solution f of the functional equation (1.1).

* It is easy to see that condition (viii) implies condition (v) given above.

3. Stability Results

The standard procedure of solving functional equations of the form (1.1) is as follows. The equation (1.1) is replaced by a related equation in which the maximization is performed over a finite range of values and in which the functions involved are evaluated only at a finite set of points. Since both of these operations in general introduce inaccuracies, we must face the question of comparing the solution of the approximate equation with the solution of the original equation. This is a problem of *stability*.

Some results concerning the stability properties for this functional equation showing how large are the changes in the solution f entailed by changes in the function q may be found in [12].

We shall present next stability theorems for equation (1.1) showing how large are the changes in the solution f entailed by changes in the transformation z .

Let us consider the functional equations

$$(3.1) \quad f(w) = \sup_{y \in Y} (q(w, y) + h(w, y) \int_X P(w, y, dx) f(z(w, y, x)))$$

and

$$(3.2) \quad F(w) = \sup_{y \in Y} (q(w, y) + h(w, y) \int_X P(w, y, dx) F(Z(w, y, x)))$$

In the following discussion, we shall assume that (3.1) and (3.2) are either of type I or of type II. According to theorems 2.1 and 2.2, there exist sequences of functions (f_n) and (F_n) converging uniformly to the unique solutions f and F of equations (3.1) and (3.2), respectively. Moreover, we shall assume that the sequence of real numbers $(\beta_n)_{n \geq 0}$ defined by

$$(3.3) \quad \beta_n = \min \{m(f_n), m(F_n)\}$$

is bounded, and we denote $\beta = \sup \{\beta_n \mid n \in \mathbb{N}\}$.

Theorem 3.1. If the functional equations (3.1) and (3.2) are of type I, then

$$(3.4) \quad \sup_{w \in W(c)} |F(w) - f(w)| \leq \beta \sum_{n=1}^{\infty} r(\alpha^n c)$$

where

$$(3.5) \quad r(c) = \sup \{\rho(Z(w, y, x), z(w, y, x)) \mid w \in W(c), y \in Y, x \in X\}.$$

Proof. From lemma 2 it follows that

$$(3.6) \quad |F_n(w) - f_n(w)| \leq \sup_{y \in Y} \{ |h(w, y)| \int_X P(w, y, dx) |F_{n-1}(Z(w, y, x)) - f_{n-1}(z(w, y, x))| \}$$

for every positive integer n , and for any $w \in W$.

For any nonnegative real number c we denote

$$(3.7) \quad r_n(c) = \sup_{w \in W(c)} |F_n(w) - f_n(w)|.$$

It is easy to see that for any $w \in W(c)$, $y \in Y$, and $x \in X$ we have $z(w, y, x) \in W(\alpha c)$ and $Z(w, y, x) \in W(\alpha c)$. Thus we obtain

$$(3.8) \quad \sup \{(|F_{n-1}(Z(w, y, x)) - f_{n-1}(Z(w, y, x))|) \mid x \in X, y \in Y, \\ w \in W(c)\} \leq \sup \{(|F_{n-1}(w) - f_{n-1}(w)|) \mid w \in W(\alpha c)\} = r_{n-1}(\alpha c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

On the other hand, from lemma 3 it follows that $f_n, F_n \in CL(W)$ for every integer $n \geq 0$. Therefore we get

$$(3.9) \quad \sup \{(|J_{n-1}(Z(w, y, x)) - f_{n-1}(z(w, y, x))|) \mid x \in X, \\ y \in Y, w \in W(c)\} \leq m(f_{n-1}) r(\alpha c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

Now, from (3.8) and (3.9) we obtain easily

$$(3.10) \quad \sup \{(|F_{n-1}(Z(w, y, x)) - f_{n-1}(z(w, y, x))|) \mid w \in W(c), y \in Y, x \in X\} \leq \\ \leq r_{n-1}(\alpha c) + m(f_{n-1}) r(\alpha c)$$

or every integer $n \geq 1$, and any $c \geq 0$.

Finally, from (3.6) and (3.10) it follows that

$$(3.11) \quad r_n(c) \leq r_{n-1}(\alpha c) + m(f_{n-1}) r(\alpha c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

In a similar manner we obtain

$$(3.12) \quad r_n(c) \leq r_{n-1}(\alpha c) + m(F_{n-1}) r(\alpha c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

From (3.3), (3.11), and (3.12) we get

$$(3.13) \quad r_n(c) \leq r_{n-1}(\alpha c) + \beta_{n-1} r(\alpha c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

Taking into account that $r_0(c) = 0$ for any $c \geq 0$, from (3.13) we obtain easily

$$(3.14) \quad r_n(c) \leq \sum_{i=1}^{n-1} \beta_i r(\alpha^{n-i} c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

The result (3.4) follows now easily from the last inequality.

Theorem 3.2. If the functional equations (3.1) and (3.2) are of type II, then

$$(3.15) \quad \sup_{w \in W(c)} |F(w) - f(w)| \leq \frac{\alpha \beta r(c)}{1 - \alpha}$$

Proof. It is easy to see that relationships (3.6) hold. Moreover, the inequalities (3.8), (3.9), and (3.10) hold, too, for $\alpha = 1$. Using these inequalities, from (3.6) we get

$$(3.16) \quad r_n(c) \leq \alpha (r_{n-1}(c) + m(f_{n-1}))$$

for every integer $n \geq 1$, and any $c \geq 0$.

In a similar manner we obtain

$$(3.17) \quad r_n(c) \leq \alpha (r_{n-1}(c) + m(F_{n-1}))$$

for every integer $n \geq 1$, and any $c \geq 0$.

Now, from (3.3), (3.16), and (3.17), we get

$$(3.18) \quad r_n(c) \leq \alpha (r_{n-1}(c) + \beta_{n-1})$$

for every integer $n \geq 1$, and any $c \geq 0$.

Taking into account that $r_0(c) = 0$ for any $c = 0$, from (3.18) we obtain easily

$$(3.19) \quad r_n(c) \leq \sum_{i=1}^{n-1} \beta_i \alpha^{n-1-i} r(c)$$

for every integer $n \geq 1$, and any $c \geq 0$.

The result (3.15) follows now easily from the last inequality.

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SCIENTIFIC LIFE OF THE FACULTY

*The National Symposium "Traian Lalescu",
Craiova, June 11–12, 1982.*

Organized under the auspices of the Romanian Society for Mathematical Sciences Department Dolj, and the University of Craiova, the symposium dedicated to the centenary of, Traian Lalescu's birth was an occasion for a wide reconsideration of the scientist, professor and patriot who became a great name in the history of Romanian culture.

A great number of speakers were reflecting in their plenary speeches quite different aspects of Traian Lalescu's personality. We thus mention some of them: Acad. N. Teodorescu — *About the scientist and promoter of the Mathematical Review*, Prof. Coleta de Sabata — *About the first Rector of the Polytechnical School from Timișoara*, Prof. emeritus Nicolae Andrei — *About the bright pupil of the „Nicolae Bălcescu” Gymnasium from Craiova*, Prof. G. Vraciu — *About one of the founders of the mathematical school in Romania*, Prof. S. Marcus — *About Traian Lalescu in the history of Romanian mathematics*, Associated Prof. Paula Diaconescu — *About the language of Traian Lalescu's texts*, Prof. Al. Zorica — *About the man and promoter*.

A great number of papers, like that of Prof. Gh. Gheorghiev from the University of Jassy were completing brightly this symposium. Some of them were bringing new elements about the cultural, scientific and social aspects of Traian Lalescu's personality. We thus mention: the papers of Marcel Țena (*The echo of a Traian Lalescu's Boundary-Value Problem*) and researchers Eliza Roman and Viorica Prodan (*Traian Lalescu — bibliographie*).

Occasioned by the „Traian Lalescu” national symposium was also organized in the Blue Hall of the University of Craiova an homagial exhibition, presenting the steps of the scientist's life, his human relations and the echo of his scientific work.

S. MARCUS

Anniversary Session dedicated to Professor Octav Onicescu.

Saturday, on October 16, 1982 had been organized, in the Hall of the Romanian Academy, the celebration of Professor Octav Onicescu, occasioned by his 90th birthday anniversary.

The speakers were: Acad. prof. Șerban Țițeica, Acad. prof. Caius Iacob, Acad. prof. Nicolae Teodorescu, prof. dr. doc. Ioan-Ioviț Popescu, corresponding member of the Romanian Academy, prof. dr. doc. Gh. Galbură, prof. dr. George Ciucu, corresponding member of the Academy, prof. dr. doc. Cabiria Andreian Cazacu, prof. dr. Ion Cuculescu, dr. doc. Marius Iosifescu, prof. dr. Mihai Demetrescu (ASE), Mircea Măciu (Director of the Scientific and Encyclopedic Publishing House). All the speakers were underlining the international prestige of Professor Onicescu's research work, his essential and world-known contribution to the theory of probability, mechanics a.s.o. These results opened new fields for a great number of Romanian and foreign researchers. Some details of his mathematical work were also stressed.

The discussions were prizing the immense contribution of Professor Onicescu to the development of Romanian science, the way he used to stimulate its progress, to fight for its prestige in the world.

A great number of scientists from very different fields may be proud of having been Professor Onicescu's students or collaborators. The recall of some important moments from their scientific evolution have shown how great the influence and the role of such a guide were.

The bright personality of the scientist and professor Onicescu is still shining. The whole audience was wishing Professor Onicescu all the best to his birthday anniversary and a further good health for new successes in his activity.

I. CUCULESCU

Anniversary Session dedicated to Professor Caius Jacob.

On March 26 and 27, 1982 a jubilee scientific session dedicated to Professor Caius Jacob, Member of the Academy of the Socialist Republic of Romania was held on the occasion of his 70th anniversary.

This session was organized by the Faculty of Mathematics of the Bucharest University, the Mathematical Sciences Department of the Academy and the Romanian Society of Mathematical Sciences.

Within the five sections, 154 papers were presented. The first section was concerned with works of general mechanics and astronomy. The second and third sections included works dealing with fluid mechanics. The fourth section was devoted to solid mechanics, and the fifth section to mathematics. All these communications were delivered at the Faculty of Mathematics on March 26. This jubilee session has constituted meanwhile an important event for the science of mechanics in our country.

A large number of academicians and professors at various universities delivered communications (The Universities of Bucharest, Iassy, Cluj-Napoca, Timișoara, Craiova, Galați and Brașov). Papers were also presented by the teaching staff of Polytechnical Institutes and other higher education institutes, by research workers at the Central Institute of Mathematics, the Institute of Hydrotechnical Research (Bucharest), The Institute of Gas and Oil (Ploiești) the Mining Institute (Petroșani), ICEPRONAV (Galați), the Centers of Astronomy in Bucharest and Cluj and by professors at secondary schools and students.

The session closed on March 27 by a special meeting at the Romanian Academy held to celebrate the 70th anniversary of professor Caius Jacob. The chairman was professor Șerban Țițeica, vice-president of the Academy, who opened the meeting.

Other speakers on this occasion were Professor Octav Onicescu, Member of the Academy, Professor Dr. Ioan Ioviț Popescu, Corresponding Member of the Academy, Rector of the Bucharest University, Professor dr. doc. Cabiria Andreian-Cazacu, Dean of the Faculty of Mathematics of the Bucharest University, Professor Nicolae Teodorescu, Member of Academy, President of the Romanian Society of Mathematical Sciences, Professor Dumitru Dumitrescu, Member of Academy, Assoc. Professor Dr. Simona Popp and Professor Dr. Doc. Lazăr Dragoș, from the Chair of Mechanics of the Faculty of Mathematics and Assoc. Professor Dr. Dorel Homentcovschi from the Bucharest Polytechnic Institute.

All the speakers underlined the rich and brilliant scientific activity carried out by Professor Caius Jacob who is one of the creators of the Romanian School of Mechanics and a remarkable cultural personality in our country.

At the end of the manifestation a comprehensive speech was delivered by Professor Caius Jacob.

SIMONA POPP

Anniversary Session dedicated to Professor Nicolae Mihăileanu.

On December 11, 1982 has been celebrated the 70th anniversary of Professor Dr. Nicolae Mihăileanu, who was teaching for over 20 years at the University of Bucharest, bringing a great contribution to the development of mathematical teaching in our country. Besides important results in differential geometry, projective geometry, noneuclidean geometry, geometry over the complex numbers, Professor Mihăileanu has also achieved a work of great value: *The History of Mathematics*, published in two volumes (1974 and 1981) at the Scientific and Encyclopedic Publishing House, Bucharest.

Occasioned by the anniversary, Professor C. Andreian-Cazacu was speaking on behalf of the Faculty of Mathematics - Bucharest, Acad. Professor Nicolae Teodorescu on behalf of the Romanian Society of Mathematicians, R. Miron on behalf of the National Committee of Geometry and Topology, D. Papuc on behalf of the teaching staff from the University of Timișoara, I. Teodorescu on behalf of the Chair of Algebra and Geometry from the Faculty of Mathematics, University of Bucharest.

C. ANDREIAN-CAZACU

Scientific Session of the Faculty of Mathematics, Bucharest, May 24-28, 1982.

The sections of the Scientific Session of our Faculty were: Algebra and Geometry, Mathematical Analysis, Mechanics, Theory of Probability and Applications, Differential Equations, Informatics and Operations Research, Applied Informatics. A number of 119 papers have been presented, out of which 30 papers belonged to students and other 25 to the researchers from the Computing Centre of the University of Bucharest.

Occasioned by the centenary of the birth of mathematician Traian Lalescu, was organized on May 28 a Symposium where Acad. Professor Nicolae Teodorescu, Acad. Professor Caius Jacob, Acad. Professor Gh. Marinescu and Professor dr. doc. Solomon Marcus were delivering speeches.

The third Colloquium on „Ordered Linear Topological Spaces”, Craiova, May 15-17, 1982

The Colloquium was organized by the staff of the scientific seminar under the guidance of Professor dr. doc. Romulus Cristescu from the University of Bucharest. The Society of Mathematical Sciences was also collaborating to this colloquium. The contributors to the colloquium were several members of the teaching staff and researchers from the Universities of Bucharest, Craiova and Brașov.

BOOKS REVIEW

I. Beju, E. Soos, P. P. Teodorescu, *Spinor and non-Euclidean tensor calculus with applications*. 275 p. Ed. tehnică, București; Abacus Press, Tunbridge Wells, Kent, 1983.

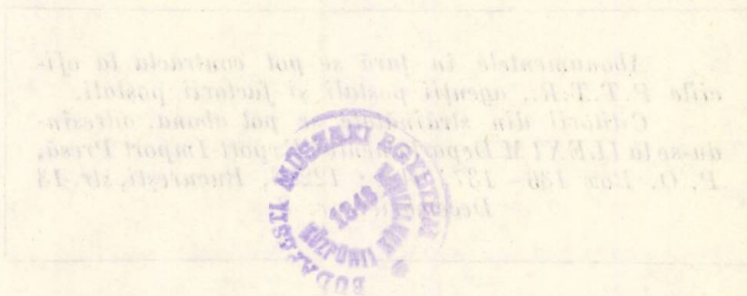
Spinor calculus and non-Euclidean tensor calculus came into being from necessities imposed by the mathematical modelling of mechanical, electromagnetical, quantum, relativistic gravitational a.s.o. phenomena. The work comprises an introductory part, devoted to the introduction of spinor calculus and of vectors and tensors on differentiable manifolds. A special attention is paid on the possibility of applying the mathematical formalism, presented in a unitary framework, in various domains of geometry, mechanics and physics. The applications presented do not exhaust the large range of problems that can occur; but taking into account the unitary character of the methods used, the reader, interested and compelled to use mathematical models, can find, without difficulty, a similar way of attacking many other problems. The connections thus appearing are sometimes unexpected; for example, the authors put into evidence the fascinating analogy between the continuous theory of dislocations and the relativistic theory of gravitation, mathematical models of domains of reality which, at the first sight, are not related in any significant way. The original character of this volume is thus emphasized.

ST. IANUȘ

I. Beju, E. Soos, P. P. Teodorescu, *Euclidean tensor calculus with applications*. 303 p. Ed. tehnică, București; Abacus Press, Tunbridge Wells, Kent, 1983.

Many physical quantities are of scalar, pseudoscalar, vectorial, pseudo-vectorial, or, in general, tensorial and pseudotensorial character. An outstanding part is played by Euclidean tensor calculus, which is presented in the titled book by using the complete orthogonal group of transformations as well as pseudo-Euclidean and Euclidean spaces where the transformations are arbitrary. Special stress is laid on the application of this mathematical formalism to various domains of mechanics (mechanics of rigid and deformable bodies and relativistic mechanics) and of physics (acoustics, optics, crystallography, electromagnetism relativistic theory of space and time), where its efficacy is fully confirmed. The work is intended for a broad audience, interested and compelled to use mathematical models; it is thus emphasized that Euclidean tensor calculus must be used not only for its elegance but also as a necessity.

ST. IANUȘ



BOOKS REVIEW

J. Hejhal, I. Jones, P. P. Teichner, Spinor and non-Euclidean tensor calculus with applications, 275 p. Ed. Technica, Bucharest; Abacus Press, Tunbridge Wells, Kent, 1983.

Spinor calculus and non-Euclidean tensor calculus came into being from necessities imposed by the mathematical modeling of mechanical, electrodynamical, quantum relativistic gravitational and other phenomena. The work comprises an introductory part, devoted to the foundation of spinor calculus and of vector and tensor or differential manifolds. A special attention is paid on the possibility of applying the mathematical formalism presented in a unified framework in various domains of geometry, mechanics and physics. The applications presented do not exhaust the large range of problems that can occur, but lead into new and the nature character of the method itself, the tensor, interested and compelled to use mathematical models, can find, without difficulty, a similar way of attacking many other problems. The connection thus appearing has sometimes unexpected; for example, the authors put into evidence the fascinating analogy between the continuous theory of deformations and the relativistic theory of gravitation. Mathematical models of domains of reality which at the first sight, are not related in any significant way. The original character of this volume is thus emphasized.

ST. IANUŞ

J. Hejhal, I. Jones, P. P. Teichner, Spinor and non-Euclidean tensor calculus with applications, 275 p. Ed. Technica, Bucharest; Abacus Press, Tunbridge Wells, Kent, 1983.

Many physical quantities are in reality, pseudoscalar, vectorial, pseudovectorial, or to speak more generally, tensorial and geometrical character. An outstanding part is played by spinor calculus, which is presented in the first book by using the tensor calculus. A group of transformations as well as pseudoscalar and Euclidean spaces where the transformations are arbitrary. Special stress is laid on the application of this mathematical formalism to various domains of mechanics (mechanics of rigid and deformable bodies and relativistic mechanics) and of physics (acoustics, optics, crystallography, electromagnetism, etc.). The work is fully contained. The work is in itself a theory of space and time, where its efficiency is fully confirmed. The work is intended for a broad audience, interested and compelled to use mathematical models. It is emphasized that Euclidean tensor calculus must be used not only for its elegance, but also as a necessity.

ST. IANUŞ

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